On periodic and chaotic behavior in a two-dimensional monopoly model

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A B S T R A C T

We study the monopoly model of T. Puu with cubic price and quadratic marginal cost functions. A numerical continuation method is used to compute branches of solutions of period 5, 10, 13 and 17 and to determine the stability regions of these solutions. General formulas for solutions of period 4 are derived analytically. We show that the solutions of period 4 are never linearly asymptotically stable. A nonlinear stability criterion is combined with basin of attraction analysis and simulation to determine the stability region of the 4-cycles. This corrects the erroneous linear stability analysis in previous studies of the model. The chaotic and periodic behavior of the monopoly model is further analyzed by computing the largest Lyapunov exponents, and this confirms the above mentioned results.

1. Introduction

During the last two decades increasing attention has been paid to the analysis of nonlinear dynamics of economic models using difference equations [1–5]. In particular, the monopoly model is well documented in [6,7,3]. Baumel and Quantdt [6] analyzed a cost-free monopoly model. They examined in both discrete and continuous systems the problem of maximizing the profit function. Puu [7] assumes a cubic price and quadratic marginal cost function when the monopoly firm maximizes the profit using as strategic variable the produced quantity. In [7] the price \( p \) for a good is represented as a monotonically decreasing function

\[
p(x) = A - Bx + Cx^2 - Dx^3,
\]

of the produced quantity \( x \). Monotonicity is implied by requiring that \( A, B, C, D > 0 \) and \( C^2 < 3BD \). The revenue of the monopolist is

\[
R(x) = p(x)x.
\]

The marginal revenue is then given by

\[
MR = \frac{dR}{dx} = p + x\frac{dp}{dx}.
\]

Also in [7], the marginal cost curve is assumed to be

\[
MC = E - 2Fx + 3Gx^2,
\]

where \( E, F \) and \( G \) are positive constants. A standard result of economic theory is that the profit is maximized at a point where \( MR = MC \). The profit function is

\[
\Pi(x) = (A - E)x - (B - F)x^2 + (C - G)x^3 - Dx^4.
\]

A simple algorithm to find the maximum of the not explicitly known function (5) is to evaluate (5) in the last two visited points \( x \) and \( y \), and use a Newton-like iteration with step size \( \delta \). We get the next point as

\[
y + \delta\frac{\Pi(y) - \Pi(x)}{y - x} = y + \delta((A - E) - (B - F)(x + y) + (C - G)(x^2 + xy + y^2) - D(x^3 + x^2y + xy^2 + y^3)).
\]

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The iteration of this procedure may lead to any of the profit maxima, to an oscillating processes, or to chaos depending on the coefficients $A$ through $G$ and the step size $\delta$. Following [7], we assume $A = 5.6, B = 2.7, C = 0.62, D = 0.05, E = 2, F = 0.3$ and $G = 0.02$. With these parameter values, the profit function (5) is symmetric about $(3, 3)$, i.e.,

$$\Pi(3 + x) = \Pi(3 - x)$$

(7)

for all $x$. The updating process (6) can be interpreted as the two dimensional mapping

$$M : \left( \begin{array}{c} x_t \\ y_t \end{array} \right) \rightarrow \left( \begin{array}{c} x_{t+1} \\ y_{t+1} \end{array} \right) = \left( \begin{array}{c} y_t + \delta P(x_t, y_t) \\ y_t \end{array} \right),$$

(8)

where

$$P(x, y) = 3.6 - 2.4(x + y) + 0.6(x^2 + xy + y^2) - 0.05(x^3 + x^2y + xy^2 + y^3).$$

(9)

The map has three steady states, which are extrema for the profit function, namely a local minimum and two local maxima. Puu [7,3] provides incomplete information on the existence of cycles of period 4 and the chaotic behavior of (8). Most of the recent literature deals with simplified versions of the Puu model (8), cf. [8–11], and none of them analyzes the dynamic behavior of the Puu model in detail.

Naimzada and Ricchiuti [10] propose to use a demand function (1) without inflection point to achieve a one-dimensional map. Their model was generalized by Askar [9] and further by Matsumoto and Szidarovszky [8]. In these models the chaotic dynamic arises via a cascade of period-doubling bifurcations.

In the present paper, we reconsider the dynamic monopoly model. The setup of this paper is as follows. In Section 2, some preliminary results are presented including corrections to the fixed points stability analysis presented in [3]. By simulations, the existence of solutions of period 4, 5, 10, 13, 17 and the chaotic behavior are investigated. In Section 3, continuation and bifurcation analysis is used to get information about the stability of 5,10,13,17-cycles under parameter variation. In all regions, further period-doubling bifurcations are found which implies the existence of orbits with higher periods as well. In Section 4, a general formula for solutions of period 4 is derived. Among other things, we discuss the symmetry properties of these solutions. The analytical stability analysis for the 4-cycles proves that they are never linearly stable. Therefore, the stability of the 4-cycle is investigated by studying the effect of small displacements applied to the eigenvector corresponding to the eigenvalue located at the stability boundary. This work, combined with simulation and the basin of attraction analysis for the 4-cycle allows us to determine the stability region of the 4-cycle. This region is larger than the one obtained in [7,3] which is based on an incorrect linear stability analysis. In Section 5, we analyze the chaotic behavior of the monopoly model by computing the largest Lyapunov exponents. This analysis confirms the results in the earlier sections.

2. Dynamic analysis by simulation

A fixed point of (8) satisfies the equations

$$x = y,$$

(10)

$$y = y + \delta P(x, y).$$

(11)

Substituting (10) into (11), we find that

$$P(x, x) = 3.6 - 4.8x + 1.8x^2 - 0.2x^3 = 0,$$

(12)

with solutions $x = 3 \pm \sqrt{3}$ and $x = 3$. In fact, $\Pi(3 \pm \sqrt{3})$ are the maxima of $\Pi(x)$ and $\Pi(3)$ is the local minimum. To determine the stability of these points, we calculate the Jacobian matrix of (8)

$$J = \begin{pmatrix} 0 & 1 \\ \frac{\partial P}{\partial y} & \frac{\partial P}{\partial x} \end{pmatrix}.$$ 

(13)

The characteristic equation is

$$P(\lambda) = \lambda^2 - \left(1 + \delta \frac{\partial P}{\partial y}\right) \lambda + \left(-\delta \frac{\partial P}{\partial x}\right) = 0.$$ 

(14)

We use the Jury test [12] to determine whether all roots of (14) lie in the open unit disk (i.e., $|\lambda| < 1$). The three conditions (Jury’s stability criterion) are given by

$$P(1) = -\delta \frac{\partial P}{\partial y} - \delta \frac{\partial P}{\partial x} > 0,$$

(15)

$$P(-1) = 2 + \delta \frac{\partial P}{\partial y} - \delta \frac{\partial P}{\partial x} > 0,$$

(16)

$$-\delta \frac{\partial P}{\partial x} < 1.$$ 

(17)

The first condition is not satisfied for $x = 3$ since

$$\left.\frac{\partial P}{\partial y}\right|_{(3,3)} = \left.\frac{\partial P}{\partial x}\right|_{(3,3)} = 0.3\delta,$$ 

hence $P(1) < 0$ for all $\delta > 0$.

Both fixed points $x = 3 \pm \sqrt{3}$ satisfy (15) and (16) since

$$\left.\frac{\partial P}{\partial y}\right|_{(3 \pm \sqrt{3},3 \pm \sqrt{3})} = \left.\frac{\partial P}{\partial x}\right|_{(3 \pm \sqrt{3},3 \pm \sqrt{3})} = -0.6\delta.$$ 

The fixed points $x = 3 \pm \sqrt{3}$ are therefore asymptotically stable if (17) is satisfied. This is the case iff $\delta \in (0, \frac{1}{2})$. For $\delta = \frac{1}{2}$ we have

$$\lambda_{1,2} = e^{\pm i\frac{\pi}{2}}.$$ 

So we have a (non-generic) 1:4 resonant
Fig. 2. (a) Two cycles of period-4 for $\delta = 2.44$. (b) Two cycles of period-17 for $\delta = 2.45$ as a time series of $x_n$ vs $n$. (c) The same cycles of period-17 in $(x_n, y_n)$-plane. (a) and (c) are point-symmetric around $(3, 3)$.

Fig. 3. (a) and (b) A chaotic attractor for $\delta = 2.62$. Note that this coexist with a 4 cycle.

Fig. 4. (a) A cycle of period-5 for $\delta = 2.63$. (b) A cycle of period-10 for $\delta = 2.7$. It obviously arises from a period-doubling of a 5-cycle.
cycles of period 4 are born. These remarkable that as we exceed the NS value be refined by a continuation method in Section 3. It is derived in [7] using a different method.

Fig. 1 plots the bifurcation diagram of (8) with \( \delta \in [0.01, 4] \). For each \( \delta \) the initial points were reset to \((x_0, y_0) = (3(\pm \sqrt{3} + \epsilon, 3 \pm \sqrt{3} + \epsilon), (3, 3))\), \( \epsilon = 10^{-5} \). 10^5 map iterations were performed and transients were discarded. This leads to a crude bifurcation diagram that will be refined by a continuation method in Section 3. It is remarkable that as we exceed the NS value \( \delta = \frac{5}{2} \) a cycle of period-4 is born. The fixed point \( x = 3 \) forms the middle line.

We see that for \( \delta \in [0, \frac{5}{2}] \), \((x_n, y_n)\) converges to a nonzero steady state and for \( \delta > \frac{5}{2} \) cycles of period 4 are born. These cycles are indicated by three upper branches and three lower branches in Fig. 1. The middle upper and lower branches in Fig. 1 are visited twice. Two typical 4-cycles are presented in Fig. 2(a) for \( \delta = 2.44 \).

In Section 4.3 we will see that the 4-cycles lose stability only for \( \delta > -\frac{5}{2} + \frac{118}{27} \approx 3.8647 \) but for \( \delta > 2.62 \) the radius of attraction is very small.

For \( \delta \) around 2.45 a stable cycle of period 17 exists, see Fig. 2(b) and (c).

For \( \delta \approx 2.62 \), the radius of convergence of the stable 4-cycle is very small and there is a nearby chaotic attractor in which a “ghost of 4-cycle” is still present, see Fig. 3(a) and (b). For \( \delta \) around 2.63, a stable cycle of period 5 exists, see Fig. 4(a). At \( \delta \) around 2.7, a stable cycle of period-10 exists, see Fig. 4(b). For \( \delta \) around 2.83 we find a stable cycle of period 13, see Fig. 5.

### 3. Analysis by numerical continuation

We perform a numerical stability analysis for the cycles of period 5, 10, 13 and 17 of (8). The stability analysis is based on a continuation method and uses the Matlab package MatContM, see [13,14].

For the cycles of period 5, 10, 13 and 17, and using the initial data in Table 1, we continue each cycle with free param-
Adding (27) to (28), we get
\[ \frac{z}{C_0} y \bigg( \frac{y}{C_0} x \bigg) = \frac{C_0}{d} \bigg( \frac{x}{C_0} z \bigg) P \bigg( \frac{y}{C_0}, \frac{y}{C_0} z \bigg). \] (31)

From (30) it now follows that \( x = y \). Plugging \( x = y \) into (23)–(26), we obtain
\[ \begin{align*}
  z - y &= \delta P(y, y), \\
  y - z &= \delta P(z, z), \\
  0 &= \delta P(y, z).
\end{align*} \] (32) (33) (34)

The set of Eqs. (32)–(34) has the solutions
\[ \begin{cases}
  (3, 3), & \text{if } \delta > 0, \\
  \left(3 \pm \sqrt{3 \delta^2 + 10 \delta}, 3 \pm \sqrt{3 \delta^2 + 10 \delta} \right), & \text{if } \delta \geq 5/3.
\end{cases} \] (35)

The solutions for which \( y = z \) are the fixed points of (8) and can be ignored. So, the general solution of period 4 for which two successive ordered pairs of cycle (18) have the same first component is given by
\[ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 \pm \sqrt{3 \delta^2 + 10 \delta} \\ \frac{x}{\delta} \\ 3 \pm \sqrt{3 \delta^2 + 10 \delta} \\ \frac{z}{\delta} \end{pmatrix}. \] (36)

Fig. 7 shows a plot of (36) for \( \delta = 2 \). Now suppose that \( x \neq y, y \neq z, z \neq w \) and \( w \neq x \) in (18). Thus, system (19)–(22) is equivalent to

Table 2
The bifurcation points on the continuation curves of the 5, 10, 13, 17-cycles with the corresponding value of \( \delta \).

<table>
<thead>
<tr>
<th>Cycle</th>
<th>LP1</th>
<th>BP1</th>
<th>PD1</th>
<th>PD2</th>
<th>PD3</th>
<th>LP2</th>
<th>PD4</th>
<th>BP2</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-Cycle</td>
<td>2.62813</td>
<td></td>
<td>2.69111</td>
<td>3.52522</td>
<td></td>
<td>3.52573</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10-Cycle</td>
<td>2.80134</td>
<td>2.69111</td>
<td>2.70485</td>
<td>2.80214</td>
<td>3.12750</td>
<td>3.12753</td>
<td>3.52149</td>
<td>3.52522</td>
</tr>
<tr>
<td>13-Cycle</td>
<td>2.47864</td>
<td>2.48136</td>
<td>2.82987</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17-Cycle</td>
<td>2.44977</td>
<td></td>
<td>2.45042</td>
<td>2.76425</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 6. Bifurcation diagram in (\( \delta, x \))-plane of period-5 (a), period-10 (b), period-13 (c) and period-17 (d) cycles.
\( (z - y)(y - x) = \delta(y - x)P(x, y), \)  \( (37) \)
\( (w - z)(z - y) = \delta(z - y)P(y, z), \)  \( (38) \)
\( (x - w)(w - z) = \delta(w - z)P(z, w), \)  \( (39) \)
\( (y - x)(x - w) = \delta(x - w)P(w, x). \)  \( (40) \)

Adding (37) to (38), (38) to (39), (39) to (40), (40) to (37), we get

\[ (z - y)(y - x + w - z) = \delta(z - x)P(x, z), \]  \( (41) \)
\[ -(w - z)(y - x + w - z) = \delta(w - y)P(y, w), \]  \( (42) \)
\[ (x - w)(y - x + w - z) = \delta(x - z)P(x, z), \]  \( (43) \)
\[ -(y - x)(y - x + w - z) = \delta(y - w)P(y, w). \]  \( (44) \)

We now distinguish two cases:

4.1.1. Case 1: \( x = z \)

If \( x = z \), we can solve system (41)–(44) for \( \{y, z, w\} \) and obtain two solutions:

\[
\begin{pmatrix}
  \frac{0.2(15\delta + r)}{\delta} \\
  \frac{-0.2(15\delta + r + 2\sqrt{15\delta^2 - 25\delta})}{\delta} \\
  \frac{0.4(15\delta + r)}{\delta}
\end{pmatrix}
\]

\[
\begin{pmatrix}
  \frac{0.2(15\delta - r)}{\delta} \\
  \frac{-0.2(15\delta - r - 2\sqrt{15\delta^2 - 25\delta})}{\delta} \\
  \frac{0.4(15\delta - r)}{\delta}
\end{pmatrix}
\]

where \( r = \sqrt{15\delta^2 + 100\delta} \). This case has been studied by Vandenameme [15]. She derived the first 4-cycle in (45).

4.1.2. Case 2: \( x \neq z \)

If \( x \neq z \), we have

\[
\begin{pmatrix}
  x - w \\
  x - z \\
  y - w \\
  y - z
\end{pmatrix}
= \begin{pmatrix}
  z \\
  z \\
  y \\
  y
\end{pmatrix}
\]

i.e.,

\[
x = w - z + y. \]

Plugging (47) for \( x \) in (41)–(44) and solving this system for \( \{y, z, w\} \), we get the another formula for two cycles of period-4:

\[
\begin{pmatrix}
  x \\
  y \\
  z \\
  w
\end{pmatrix}
= \begin{pmatrix}
  \frac{27\delta^2 - 120\delta + \frac{4}{5}s\sqrt{\delta(20s + 75\delta)}}{9\delta - 40}\delta \\
  \frac{27\delta^2 - 120\delta + \frac{2}{5}s\sqrt{\delta(20s + 75\delta)}}{9\delta - 40}\delta \\
  \frac{3 + \frac{1}{5}\sqrt{\delta(20s + 75\delta)}}{y}
\end{pmatrix}
\]

Table 3

<table>
<thead>
<tr>
<th>Cycle</th>
<th>Stability region</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-Cycle</td>
<td>((2.62813, 2.69111) \cup (3.52522, 3.52573))</td>
</tr>
<tr>
<td>10-Cycle</td>
<td>((2.69817, 2.70485) \cup (2.80134, 2.80214))</td>
</tr>
<tr>
<td>13-Cycle</td>
<td>((2.47864, 2.48136) \cup (2.82987, 2.83005))</td>
</tr>
<tr>
<td>17-Cycle</td>
<td>((2.44977, 2.45042) \cup (2.76425, 2.76432))</td>
</tr>
</tbody>
</table>

where \( s = \sqrt{9\delta^2 + 45\delta - 100} \).

4.2. The symmetry property

The 4-cycles (36) (the green points), (45) (the blue points) and (48) (the red points) are shown in Fig. 8. The 4-cycles (45) and (48) clearly differ only in the choice of the first point of the cycle. Fig. 8 possesses a point symmetry around \((3, 3)\) i.e., the upper (red/blue) 4-cycle is obtained by a rotation of the lower (red/blue) one over 180°.

The vertices of each 4-cycle form a perfect square. The points \(c_1 = \left(\frac{0.2(15\delta + r)}{\delta}, \frac{0.2(15\delta - r)}{\delta}\right)\) and \(c_2 = \left(\frac{0.2(15\delta + r)}{\delta}, \frac{0.2(15\delta - r)}{\delta}\right)\) are the center points of the upper and lower square respectively; \(c = (3, 3)\) is the center point for the big square.

4.3. Stability analysis

The next step is to determine the stability of the cycles in Section 4.1.
\[
\begin{pmatrix}
\frac{27\delta^2 - 120\delta - \frac{2}{5} \sqrt{\delta(20s + 75\delta)} + 3 \sqrt{\delta(20s + 75\delta)}\delta}{(90 - 40\delta)} \\
\frac{27\delta^2 - 120\delta - \frac{2}{5} \sqrt{\delta(20s + 75\delta)} + \frac{3}{5} \sqrt{\delta(20s + 75\delta)}\delta + 4 \sqrt{\delta(20s + 75\delta)}}{(90 - 40\delta)} \\
3 - \frac{1}{5} \frac{\sqrt{\delta(20s + 75\delta)}}{\delta}
\end{pmatrix},
\]

(48)

Consider the 4-cycle (36). The Jacobian of the 4-cycle is the product of the Jacobians evaluated at each point of the cycle (36), i.e.,

\[J_{41} = J(w, x)J(z, w)J(y, z)J(x, y),\]

(49)

where

\[J(x, y) = \begin{pmatrix} 0 & 1 \\ \frac{\partial}{\partial x} P(x, y) & 1 + \frac{\partial}{\partial y} P(x, y) \end{pmatrix}\]

and the other Jacobians follow by cyclicity.

After some computations we find

\[J_{41} = \begin{pmatrix} 0.36\delta^2 + 3.6\delta + 9 & 0.36\delta^2 + 3.6\delta + 8 \\ 0 & 1 \end{pmatrix},\]

(50)

with eigenvalues

\[\lambda_{1, 2} = \{1, 0.36\delta^2 + 3.6\delta + 9\}.\]

(51)

The second eigenvalue is always greater than 1 for \(\delta \geq \frac{1}{2}\). So the 4-cycle system (36) is unstable.

On the other hand, for the first 4-cycle in (45) (the same results hold for the second 4-cycle) the Jacobian matrix is given by

\[J_{42} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},\]

(52)

where

\[A = \frac{144}{15625} \sigma\psi\delta^2 + \frac{36}{3125} \sigma^2 - \frac{216}{15625} \sigma^3 - \frac{348}{625} \sigma^4 - \frac{84}{3125} \sigma^5,\]

\[B = \frac{121}{15625} \sigma\psi\delta^2 + \frac{36}{3125} \sigma^2 - \frac{276}{3125} \sigma^3 - \frac{36}{25} \sigma^4 + \frac{116}{625} \sigma^5 + \frac{16}{5},\]

\[C = \frac{432}{390625} \sigma\psi\delta^2 + \frac{3024}{78125} \sigma^2 + \frac{2592}{78125} \sigma^3 - \frac{252}{15625} \sigma^4,\]

\[D = \frac{3888}{390625} \sigma^4 + \frac{8856}{78125} \sigma^3 + \frac{144}{3125} \sigma^2 - \frac{3348}{625} \sigma^4 - \frac{3553}{3125} \sigma^5,\]

\[\sigma = \sqrt{3\delta^2 + 20\delta},\]

\[\psi = \sqrt{3\delta^2 - 5\delta}.\]
The characteristic equation is
\[ P_{4,2}(\lambda) = \lambda^2 - (k + 1)\lambda + k = 0. \]  \hspace{1cm} (53)
where
\[ k = \frac{3888}{390625} d^4 + \frac{7776}{78125} d^3 - \frac{3168}{15625} d^2 - \frac{3056}{3125} d + \frac{2993}{625}. \]

At any point \((\xi, \eta)\) of the 4-cycle (45), there are two eigenvalues
\[ \lambda_{1,2} = (1, k). \]  \hspace{1cm} (54)
From (54) we infer the following results on the stability and bifurcations of the 4-cycle system (45):

- For \( \frac{2}{3} < \delta < \frac{2}{3} + \frac{5}{9} \sqrt{21} \), there are two eigenvalues
  \[ \lambda_1 = 1 \text{ and } |\lambda_2| < 1. \]
- At \( \delta = \frac{5}{3} + \frac{25}{78} \sqrt{21} \) a 1:1 resonant NS bifurcation occurs at
  \[ (x, y) = \left( \frac{3 + 3\sqrt{26 + 2\sqrt{21}}}{-9 + 5\sqrt{21}}, 3 \right. \]
  \[ + \left. \frac{3\sqrt{26 + 2\sqrt{21}} - 12\sqrt{11 - 2\sqrt{21}}}{-9 + 5\sqrt{21}} \right). \]
- For \( \delta > \frac{2}{3} + \frac{5}{9} \sqrt{21} \), there are two real eigenvalues
  \[ \lambda_1 = 1 \text{ and } |\lambda_2| > 1, \]
and hence, the 4-cycle is unstable.

Since there is always an eigenvalue 1, the 4-cycle is never linearly asymptotically stable. The stability and the existence of bifurcation points for \( \frac{2}{3} < \delta < \frac{2}{3} + \frac{5}{9} \sqrt{21} \) are determined by the stability analysis in the direction of the eigenvector corresponding to \( \lambda_1 = 1 \).

Let \((x, y)^T\) be a fixed point of the fourth iterate of (8). For \( 0 < \epsilon \ll 1 \), let \( \epsilon v^T \) be a small perturbation of \((x, y)^T\) where \( v \) is the right unit eigenvector corresponding to the eigenvalue 1. We decompose
\[ M^4(x, y)^T + \epsilon v^T = \alpha v + \beta w + (x, y)^T, \]  \hspace{1cm} (55)
where \( \alpha, \beta \) are scalars and \( w \) is an eigenvector corresponding to \( \lambda_2 \).

Taking inner products of (55) with the left eigenvector \( v^T \) corresponding to the eigenvalue 1, we get
\[ \chi_\epsilon = \frac{(v^T)[M^4((x, y)^T + \epsilon v^T) - (x, y)^T]}{(v^T)v}. \]  \hspace{1cm} (56)
In the sense of the definition of the stability on a specific eigenvector of the linearized system at the fixed point (see for example [16, Chapter 2]), the 4-cycle at the fixed point \((x, y)^T\) is stable if \( |\chi_\epsilon| < 1 \) for all sufficiently small \( \epsilon \). Moreover, the fixed point \((x, y)^T\) of the 4-cycle is unstable in the direction of \( v \) if \( |\chi_\epsilon| > 1 \) for arbitrarily small values of \( \epsilon \).

The vectors \( v \) and \( v^T \) are given by:
\[ v = \begin{pmatrix} -375 \delta^2 + 18 \delta \sigma \phi + 2500 \delta^2 + 145 \sigma \phi \\ 8 \sigma \phi^2 + 7800 \phi - 87 \delta^2 + 260 \delta \phi \\ 5 \end{pmatrix}, \]  \hspace{1cm} (57)
\[ v^T = \begin{pmatrix} 10 \delta \sigma \phi^2 + 216 \delta^2 - 54 \delta \sigma \phi + 212 \delta^3 - 11 \sigma \phi + 87 \delta^2 + 260 \delta \phi \\ 18 \delta^3 + 15 \delta^2 + 12 \sigma \phi - 700 \phi + 55 \sigma \phi \\ 1 \end{pmatrix}. \]  \hspace{1cm} (58)
Using Matlab, we compute numerically the value \( \chi_\epsilon \) for a large number of values of \( \delta \in [\frac{2}{3}, 5] \) (33,324 points, uniformly distributed) for \( \epsilon = 10^{-2} \) and \( 10^{-6} \). The results are presented in Fig. 9. The 4-cycles where \( |\chi_\epsilon| < 1 \) are plotted in green. The 4-cycles where \( |\chi_\epsilon| > 1 \) are plotted in blue. The first labeled points are those where \( |\chi_\epsilon| > 1 \) becomes large for \( \delta < \frac{2}{3} + \frac{5}{9} \sqrt{21} \). The second labeled points indicate the 1:1 resonant NS point for which \( \delta = \frac{2}{3} + \frac{5}{9} \sqrt{21} \).

We see that the change of sign happens for increasing values of \( \delta \) if \( \epsilon \) tends to zero. By numerical simulation for a large number of initial points computed by (45) for different values of \( \delta \) in the range \([2.71, 3.9]\) we find that the 4-cycle is stable for all values of \( \delta \) smaller than the value of the bifurcation point (i.e., the 1:1 resonant NS point) but with a very small domain of attraction. Fig. 10(b) shows what happens if we round the initial point in Fig. 10(a) to 8 digits: the initial point is no longer in the domain of attraction of the 4-cycle. For all \( \delta \) greater than the value of the bifurcation point the 4-cycle is unstable, see Fig. 10(c).
Fig. 10. (a) Stable 4-cycle for $\delta = 3.85$ where the initial point is computed by (45), exact to machine precision. (b) Behaviour if the initial point in (a) is rounded to 8 digits: the point is no longer in the domain of attraction of the 4-cycle. (c) Unstable 4-cycle for $\delta = 3.88$ with initial point exact to machine precision.

Fig. 11. The basins of attraction of the 4-cycles (45) (a) for $\delta = 2.6$; (b) for $\delta = 2.62$; (c) for $\delta = 2.615$ and the initial points located in the range $[4.36, 4.56] \times [3.42, 3.62]$ and (d) for $\delta = 2.616$ and the initial points located in the same range as in (c).
To further corroborate this result, we explore the basin of attraction of the 4-cycles (45) by performing $10^5$ map iteration at 40,000 different initial points located in the range $[-1.7] \times [-1.7]$. Fig. 11 shows the basins of attraction of the 4-cycles (45) for four values of $\delta$. The points in the attraction domain of the first 4-cycle in (45) are colored red, in the second 4-cycle green. The yellow points are those where no convergence was established after $10^5$ iterations.

For $\delta \leq 2.6$ the basin of attraction is connected, then it shrinks and contains holes. Already for $\delta = 2.615$ there are points very close to the 4-cycle which are not in its domain of attraction. However, numerical simulations show that even for values of $\delta$ slightly smaller than $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \sqrt{2}$ the 4-cycle has a small radius of attraction, which is not the case for values slightly larger than $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \sqrt{2}$. So the loss of stability of the 4-cycle is, in fact, caused by a 1:1 resonant Neimark–Sacker bifurcation.

5. Analysis by Lyapunov exponents

A quantitative measure of chaos is the Lyapunov exponent ($\lambda$), which measures the exponential separation of nearby orbits. A positive Lyapunov exponent can be considered as an indicator of chaos.

We apply the procedure described in [17,18] to calculate the largest Lyapunov exponent for $\delta \in [0.01, 4]$. For each $\delta$ the initial point is reset to $(x_0, y_0) = (3 + \sqrt{3} - \epsilon, 3 + \sqrt{3} - \delta)$, $\epsilon = 10^{-5}$ and $10^5$ map iterations are performed. The end result is presented in Fig. 12(a) and enlarged in Fig. 12(b). Comparing Fig. 12 with the bifurcation diagram as presented in Fig. 1 and to the cycle diagrams (Figs. 2–5), we notice that:

1. For $\delta \in [0.01, \frac{3}{2}]$, $\lambda$ is negative; it approaches zero at the 1:4 resonant NS bifurcation point $\delta = \frac{3}{4}$ where the cycles of period 4 are born. The largest Lyapunov exponent remains zero for $\frac{3}{4} < \delta < 2.615$ except for the dip around $\delta = 2.45$ caused by a stable cycle of period 17.
2. The first rise for $\lambda$ is around $\delta = 2.62$ due to the chaotic behavior which is clear in Fig. 3(a) and (b).
3. The second dip between $\delta = 2.62$ and $\delta = 2.7$ is due to the period 5 behavior which is clear in Fig. 4(a). As $\delta$ touches zero around $\delta = 2.7$ a cycle of period 10 is born.
4. For $\delta > 2.7$, the largest Lyapunov exponent increases and the system becomes more and more chaotic, except for the big dip caused by the period 13 behavior around $\delta = 2.83$, see also Fig. 5.

The chaotic behavior of the monopoly model can now be better understood with the help of the greatest Lyapunov exponents analysis discussed above. Roughly speaking, the positive Lyapunov exponent for $\delta \in (\approx 2.62) \cup (2.7, \infty) \setminus \{\approx 2.83\}$ confirms the predominance of a chaotic attractor.

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