Surface properties for rarefied circular jet impingement on a flat plate

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In this paper, we investigate rarefied jet gas flows out of a circular exit impinging on a vertical flat plate. We employ a constraint relation about the velocity components of gas particles leaving a nozzle exit point and arriving at a given spatial point outside the nozzle. This relation leads to several analytical expressions for collisionless flow property distributions on the plate surface, including density, slip-velocity, temperature, pressure, shear stress, and heat flux. Numerical simulation results obtained with the direct simulation Monte Carlo method validate the analytical collisionless flow solutions. The impingement properties on the plate surface are accurate when the Knudsen number is large. © 2011 American Institute of Physics. [doi:10.1063/1.3549934]

I. INTRODUCTION

Jet impingement flows can be found in the cooling of hot metal, plastic, glass sheets, electronics, drying paper, fabric, and other applications. Rarefied circular jet flows impinging on a flat plate have many important applications as well. Several examples include molecular beam, plasma plumes from electric propulsion devices, thin film deposition process inside a vacuum chamber,1 and rocket plume impingement on spacecraft solar panels.2 Among those examples, we are interested in plate surface properties. In particular, a better understanding of the pressure and shear stress distributions on the plate can benefit some applications such as (1) computing the drag force on spacecraft solar panel surfaces and (2) determining the lunar ground breaking up position and the beginning position of sand lifting up due to rocket plume impingement.

Usually a rarefied plume jet is modeled by assuming free molecular flows with a nonzero, uniform average exit velocity \( U_0 \). Woronowicz3 presented a set of analytic point source to model behaviors ranging from molecular effusions to rocket plumes. Liepmann4 reported the efflux of gases through circular apertures, which is an example of a transition from gas dynamics to gas kinetic regime. Narasimha5 obtained the exact solutions of density and velocity distributions for a free molecular effusion flow. Brook6 reported the density field of free molecular flow from an annulus to study the gas leakage effect from a spacecraft hatch. Recently, Lilly et al.7 reported their work on measurement and computation of mass flow and momentum flux through short tubes for rarefied gas. For the case of free molecular flows with a nonzero average velocity, usually the problems are complicated.8 There are some numerical and experimental results in the literature for the plume impingement problem. Maddox9 reported a computation method to determine the drag and heat flux by expanded plumes to adjacent surfaces. Kannenberg and Boyd10 proposed some plume impingement results on a plate surface with a hyperthermal flow limit assumption. Wang and Li11,12 numerically analyzed the performance of microelectromechanical systems-based nozzles at moderate and low temperatures using the direct simulation Monte Carlo (DSMC) method.13 The predictions were compared with existing experimental results and continuum theory. Chen14 analyzed the impact force acting on a flat plate which is exposed normally to a rarefied plasma plume issuing from a thruster with an annular or circular exit section at the free-molecule flow regime. In a previous study,15 some detailed macroscopic solutions of collisionless plume flows were provided, and they serve as the foundation for our study of this problem.

For a lunar or Martian landing mission with a retro-rocket, determining the interactions among rarefied rocket plume, crater, and dust is one of the most challenging tasks. Performing an experimental study with accurate parameters is almost impossible due to the special environment; as such, we have to rely on numerical simulations and analytical studies to aid our understanding of this problem. At the beginning of this challenging task, recently we started with the problem of a two-dimensional rarefied gas jet impingement on a flat plate surface.16 In this paper, we continue our work on the problem of a rarefied gas jet out of a three-dimensional circular jet.

This paper is organized as follows. Section II describes the problem and provides analysis and exact solutions; Sec. III provides simplified solutions; Sec. IV provides some validations by comparing the analytical results with several particle simulation results; and Sec. V summarizes this study.

II. RAREFIED JET IMPINGING ON A FLAT PLATE

This section discusses the problem of a rarefied circular jet impinging on a flat plate; in particular, it concentrates on the case of collisionless flow situation. Besides the macroscopic flowfield properties, we are specifically interested in the plate surface properties. Based on these analytical results, we can provide fair estimations for less rarefied flow situations, e.g., with Knudsen number close to 1.0.
A rarefied jet plume, with known number density \( n_0 \), average velocity \( U_0 \), and temperature \( T_0 \), is fired from a circular nozzle of a diameter \( D \); one plate is placed at a distance of \( L \) from the nozzle; the plate may extend widely or even to infinity for specific applications. Figure 1 illustrates the problem under consideration. In this situation, the contribution to the analytical solutions of the number density and velocities at any point \((X, 0, Z)\) in front of the nozzle were derived in\(^\text{15}\)

\[
n_1(X, 0, Z)/n_0 = \exp(-S_0) \int_{-\pi/2}^{\pi/2} d\theta \int_0^{r'} rK dr, \tag{4}
\]

\[
U_1(X, 0, Z) = \bar{\beta}_0 = \exp(-S_0) n_0 \int_{-\pi/2}^{\pi/2} d\theta \int_0^{r'} rM dr, \tag{5}
\]

\[
W_1(X, 0, Z) = \bar{\beta}_0 = \exp(-S_0) n_0 \int_{-\pi/2}^{\pi/2} d\theta \int_0^{r'} (Z - r \sin \theta) rM dr, \tag{6}
\]

where

\[
Q(r, \phi, Z) = \frac{X^2}{X^2 + r^2 + Z^2 - 2rZ \sin \theta} = \cos^2 \phi \sum_{n=0}^{\infty} P_n(\sin \phi \sin \theta) \times \left( \frac{r}{X^2 + Z^2} \right)^n. \tag{7}
\]

\[
K = Q \left[ QS_0 + \left( \frac{1}{2} + QS_0^2 \right) \sqrt{\frac{\pi}{Q}} \right] \times \left[ 1 + \text{erf}(S_0 \sqrt{Q}) \right] \exp(S_0^2 Q), \tag{8}
\]

\[
M = Q^2 \left[ QS_0^2 + 1 + S_0 \left( \frac{3}{2} + QS_0^2 \right) \sqrt{\pi Q} \right] \times \left[ 1 + \text{erf}(S_0 \sqrt{Q}) \right] \exp(S_0^2 Q), \tag{9}
\]

where \( \beta_0 = 1/(2RT_0) \); \( R \) is the gas constant; \( P_n(\sin \phi \sin \theta) \) are the Legendre polynomials; \( \phi = \arctan(Z/X) \); the subscript \( 0 \) represents the nozzle exit; \( 1 \) represents the free plume; and \( S_0 = U_0 / \sqrt{2RT_0} \) represents the plume exiting speed ratio. More details and validation work about \( n_1, U_1, \) and \( V_1 \) solutions are available in the previous study\(^\text{15}\). For convenience, we present the results in this paper. Further, we provide the temperature field solution as follows:
\[ T_1(X,0,Z)/T_0 = \frac{U^2 + W^2}{3RT_0} + \frac{4 \exp(-S_0^2) n_0}{3 \pi^2 X^2} \int_{-\pi/2}^{\pi/2} d\theta \int_0^{R'} \frac{N}{Q} r dr, \]  

where \( N \) is

\[ N = S_0 Q \left[ \frac{5}{4} + \frac{Q S_0^2}{2} \right] + \frac{1}{2} Q^3 \pi \left[ \frac{3}{4} + 3 Q S_0^2 + Q^2 S_0^4 \right] \times [1 + \text{erf}(S_0 \sqrt{Q})] \exp(S_0^2 Q). \]  

The pressure expression is available from the equation of state with known \( n_1(X,0,Z) \) and \( T_1(X,0,Z) \).

For plume impingement on the vertical flat plate, the number density on the plate consists of two terms,

\[ n_2(X,0,Z) = n_1(X,0,Z) + n'_w(X,0,Z) = n_1(X,0,Z) [1 + U_1(X,0,Z) \sqrt{\pi \beta_w}], \]  

where the subscript \( w \) represents the wall, \( \beta_w = 1/(2RT_w) \), and the left hand side term \( n_2(X,0,Z) \) is the density solution for the plume impingement problem. At the right hand side, \( n'_w(X,0,Z) \) is the new density factor contributed from the plate where the derivation of \( n'_w(X,0,Z) \) is the same as that for the two-dimensional case presented in Ref. 16, and \( n_1(X,0,Z) \) is the free plume solution directly evaluated from Eq. (4).

The slip-velocity at the plate is obtained with the following relation:

\[ W_2(X,0,Z) \sqrt{\beta_0} = \frac{2 \exp(-S_0^2)}{X \sqrt{\pi}} \frac{n_0}{n_2} \int_{-\pi/2}^{\pi/2} d\theta \int_0^{R'} G(Z - r \sin \theta) r dr, \]  

where \( G \) is

\[ G = \frac{Q^2}{2} \left[ S_0 \sqrt{Q} \left( \frac{3}{2} + Q S_0^2 \right) \exp(S_0^2 Q) [1 + \text{erf}(S_0 \sqrt{Q})] + Q S_0^2 + 1 \right]. \]  

This slip-velocity is important for simulations of dust particle-gas flow with the discrete element method (DEM)\(^{17,18}\) because it provides the crucial input data. Since the plate normal direction is defined as \((-1, 0, 0)\) at the plate surface, we can obtain other properties such as temperature, pressure, and shear stress on the plate with simpler formats,

\[ T_2(X,0,Z)/T_0 = \frac{n_w}{2 n_2(X,0,Z) \epsilon} + \frac{4 \exp(-S_0^2) n_0}{X \sqrt{\pi}} \int_{-\pi/2}^{\pi/2} d\theta \int_0^{R'} N r dr, \]  

where \( \epsilon = T_0/T_w \). The pressure expression on the wall is available from the equation of state with known \( n_2(X,0,Z) \) and \( T_2(X,0,Z) \). Shear stress on the plate surface has the following final analytical format:

\[ \frac{\tau_{zz}}{\rho U_0^2/2} = 4 \exp(-S_0^2) \int_{-\pi/2}^{\pi/2} d\theta \int_0^{R'} r \left[ N / X \right] [Z - r \sin \theta - \sqrt{\beta_0 W_2 G}] dr. \]

The heat flux on the plate surface is

\[ \frac{q}{\rho U_0^2/2} = 2 \int_{-\pi/2}^{\pi/2} \int_0^{R'} [J + \beta_0 W_2 G] dr \]

\[ - \frac{1}{2 S_0^2 n_0} \sqrt{\frac{1}{\pi \epsilon}} (2 + W_2^2 \beta_0), \]

where the first term is the heat flux which is contributed by the incident particles from the nozzle, and the second term is the heat flux related to the reflected particles. \( J \) is

\[ J = \frac{Q^2}{2} \left[ S_0 Q \left( S_0 Q + \frac{9}{2} \right) + 2 \right] + S_0 Q^2 \pi \]

\[ \times \left( \frac{15}{4} + 5 Q S_0^2 + Q^2 S_0^4 \right) [1 + \text{erf}(S_0 \sqrt{Q})] \exp(Q S_0^2). \]

### III. SIMPLIFIED ANALYTICAL RESULTS

Because the previous solutions contain integrals, it is necessary to utilize a complex code to evaluate them. To further simplify the results, we utilize a simplified formula at far field for \( Q \),

\[ Q = \cos^2 \phi \left[ P_0 (\sin \phi \sin \theta) + P_1 (\sin \phi \sin \theta) \right] \]

\[ \times \left( \frac{r}{\sqrt{X^2 + Z^2}} \right)^2 \approx \cos^2 \phi, \]

which means that \( Q \) is a function of \( X \) and \( Z \) only, or \( D/L \ll 1 \); for many applications, such a simplification is reasonable. As such, the simplified analytical solutions are
FIG. 3. Profiles of normalized number density, $n_1(X,0,Z)/n_0$, $S_0=2$, $L/D=10$, collisionless.

$$ n_1(X,0,Z)/n_0 = \frac{K}{2\sqrt{\pi}} \left( \frac{R'}{X} \right)^2 \exp(-S_0^2), $$

$$ U_1(X,0,Z)/\sqrt{\beta_0} = \frac{M}{2\sqrt{\pi}n_1} \left( \frac{R'}{X} \right)^2 \exp(-S_0^2) = \frac{M}{K}, $$

$$ W_1(X,0,Z)/\sqrt{\beta_0} = \frac{M}{2\sqrt{\pi}n_1} \left( \frac{ZR'^2}{X^3} \right) \exp(-S_0^2) = \frac{M}{K \sqrt{\pi} X}, $$

$$ T_1(X,0,Z)/T_0 = -\frac{2M^2}{3K^2} \left( 1 + \frac{Z^2}{2X^2} \right) + \frac{2N}{3Q\sqrt{\pi}} \left( \frac{R'}{X} \right)^2 \frac{n_0}{n_1} \exp(-S_0^2), $$

$$ n_2(X,0,Z)/n_0 = \frac{K}{2\sqrt{\pi}} \left( \frac{R'}{X} \right)^2 \left( 1 + \frac{M}{K \sqrt{\pi} \epsilon \beta} \right) \exp(-S_0^2), $$

FIG. 4. Profiles of normalized slip-velocity, $W_2(X,0,Z)/\sqrt{2RT_w}$, $S_0=2$, $L/D=10$, collisionless.

$$ W_2(X,0,Z)/\sqrt{\beta_0} = \frac{G}{V^2} \frac{n_0 Z R'^2}{n_2 X^3} \exp(-S_0^2) = \frac{2G}{(K + M \sqrt{\pi} \epsilon) X} Z, $$

$$ T_2(X,0,Z)/T_0 = \frac{n_{nw}}{2en_2(X,0,Z)} + \frac{2N n_0}{\sqrt{\pi} n_2} \left( \frac{R'}{X} \right)^2 \exp(-S_0^2) $$

$$ = \frac{n_{nw}}{2n_2(X,0,Z)\epsilon} + \frac{4N}{K + M \sqrt{\pi} \epsilon}, $$

$$ \frac{\tau_{es}}{\rho U_0'^2} = \frac{2}{S_0^2 \sqrt{\pi}} \left[ \frac{R'}{X} \right]^2 \left[ N^{-1} X - \beta_0 W_2 G \right] \exp(-S_0^2), $$

$$ \frac{q}{\rho U_0'^2} = \frac{1}{S_0^3 \sqrt{\pi}} \left[ \frac{R'}{X} \right]^2 \left[ J + \beta_0 W_2 G \right] \exp(-S_0^2) $$

$$ - \frac{1}{2S_0^3 n_0} \sqrt{\frac{1}{\pi \epsilon}} (2 + \beta_0 W_2 G). $$

As we can see, the nozzle exit radius $R'$ is included in $n_2$, $T_2$, $W_2$.  

FIG. 5. Profiles of normalized pressure on the plate, $P(X,0,Z)/(\rho_0 U_0'^2)$, $S_0=2$, $L/D=10$, collisionless.

FIG. 6. Comparison of normalized pressure on the plate for different Kn number, $S_0=2$, $L/D=10$. 

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As such, these simplifications shall be more accurate than a single point source treatment and more convincing for coding.

IV. VALIDATIONS

In this section, we validate the above analytical results with a specific particle simulation package called GRASP, which was developed at New Mexico State University. We simulate a rarefied argon jet impinging on a plate with different $S_0$ and Kn using an axisymmetric configuration, the nozzle diameter $D=0.2$ m, and height $X=2.0$ m.

Figure 2 shows the pressure contours for a free plume flow without a plate. The exact analytical solutions and the numerical ones are virtually identical. Because the pressure results involve $n_1$, $U_1$, $V_1$, and $T_1$, the perfect agreement between the DSMC and the analytical results indicates that all other analytical expressions are valid as well.

For the near plate or on plate properties, we are especially concerned with the slip-velocity, pressure, shear stress, and heat flux distributions along the plate. For most plume impingement applications, the pressure and shear stress distributions are important. For a DEM simulation of lunar dust flow, the ground slip-velocity distribution is another necessary input variable. Figure 3 shows the normalized number density profiles near the plate for collisionless flow with different values of $S_0$. The analytical treatment, the simplified solution, and the DSMC simulation have consistent results.

Figure 4 shows the normalized slip-velocity profiles along the plate for collisionless flow with different $S_0$. As $S_0$ increases, the values of the slip-velocity increase. The analytical result and the simplified solution are consistent with the DSMC results for the same value of $S_0$ with minor discrepancies.

Figure 5 shows the normalized pressure on the plate surface for Kn=100 with different values of $S_0$. Since the plate normal direction is determined, the pressure for this situation only includes the components related to the normal direction. The analytical treatment, the simplified solution, and the DSMC simulation have consistent results, and as $S_0$ increases, the value of the maximum pressure on the plate increases. Figure 6 shows a comparison of the analytical nor-

\[ \frac{\tau_w}{0.5 \rho U_0^2} \]

\[ \frac{q_w}{0.5 \rho U_0^3} \]

\[ Z/D \]

\[ \tau_w/(0.5 \rho U_0^2) \]

\[ q_w/(0.5 \rho U_0^3) \]

\[ S_0 = 2, X/D = 10, \text{Kn}=100 \]

\[ S_0 = 2, X/D = 10, \text{Kn}=1 \]

\[ S_0 = 3, X/D = 10, \text{Kn}=50 \]

\[ \text{DSMC, } S_0 = 2, X/D = 10, \text{Kn}=100 \]

\[ \text{Analytical, } S_0 = 1, X/D = 10 \]

\[ \text{Analytical, } S_0 = 2, X/D = 10 \]

\[ \text{Analytical, } S_0 = 3, X/D = 10 \]

\[ \text{Simplified Eqn., } S_0 = 2 \]

\[ \text{FIG. 8. Comparison of normalized shear stress on the plate for different Kn number, } S_0=2, L/D=10. \]

\[ \text{FIG. 9. Comparison of normalized heat flux on the plate for different Kn number, } S_0=2, L/D=10. \]
Figure 9 shows the normalized heat flux for $\text{Kn}=\infty$ with different values of $S_0$. The analytical treatment, the simplified solution, and the DSMC simulation have consistent results. Figure 10 shows a comparison of the analytical solution of the heat flux with the DSMC for different $\text{Kn}$ number and $S_0=2$. There are no significant variations in the results.

Figure 11 shows the normalized maximum shear stress on the plate with different $\text{Kn}$ numbers. We conclude that as the $\text{Kn}$ number increases, the maximum shear stress from the DSMC becomes closer to the analytical result of Eq. (16). There is a minor difference in the maximum shear stress values of the analytical and simplified formula. Figure 12 shows the normalized critical location of the maximum shear stress on the plate for different $\text{Kn}$ numbers. It is evident that as the $\text{Kn}$ number increases, the location of the maximum shear stress on the plate becomes closer to the analytical value. There is a minor difference in the location for the maximum shear stress on the plate from the analytical and the simplified formulas.

V. SUMMARY

We have analyzed the problem of a rarefied circular jet flow impinging on a flat plate and validated the results with several DSMC simulations. First, we revisited the complete solutions of collisionless free plume expanding into vacuum from a circular exit. By adopting a velocity-direction and geometry relation, the flowfield density, velocity distributions, temperature, and pressure were determined. These results are helpful to study less rarefied free plume flows and are the foundations for the plume impingement problem. By adding a plate, we derived some surface properties such as slip-velocity, pressure, shear stress, and heat flux. For collisionless flows, the analytical results are virtually identical to the DSMC simulation results. For the large range of rarefied regime with $\text{Kn}=1$, we can adopt the analytical results presented in this paper for fast engineering estimations with minor discrepancies. Even though the analytical results are complex, the evaluation speed is faster than particle simulations.

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