Cost driven traffic assignment in transportation networks

Doraid Dalalah*

Faculty of Engineering,
Industrial Engineering Department,
Jordan University of Science and Technology,
P.O. Box 3030, Irbid, 22110, Jordan
E-mail: doraid@just.edu.jo
*Corresponding author

Wasfi A. Al-Rawabdeh

Faculty of Economics and Administrative Sciences,
Department of Management,
The Hashemite University,
P.O. Box 150459, Zarqa, 13115, Jordan
E-mail: rawabdeh@hu.edu.jo

Abstract: A cost driven traffic assignment model is proposed to maximise the global benefit of transportation network users. A decentralised primal-dual time synchronised approach is used to solve for the optimal traffic rates subjecting the objective function to the bounds of the network tolerable costs. For a successful operation of the transportation network, the affordable upper bounds on the costs have to be set carefully. A breakeven analysis was presented to estimate the network charges. It came across that the longer the route, the higher its charge. The breakeven surcharges represent the minimum quantities a network may charge from its commuters for a profitable operation. The presented scheme demonstrates a great control on the queue sizes and motivates resource allocation in heterogeneous transportation networks. The experimental results reveal an attractive convergence, performance and sensible resource assignment between the network users while achieving comprehensive optimality in a distributed approach.

Keywords: traffic; transportation; networks; optimisation; breakeven analysis.


Biographical notes: Doraid Dalalah received his BSc in Mechanical Engineering from Jordan University of Science and Technology, 1996. He received his Master in Industrial Engineering from Jordan University in 1999. He finished his PhD degree from Lehigh University-USA in Industrial and System Engineering, 2005. Since then, he has been working as an Assistant Professor at Jordan University of Science and Technology. His research interests include parallel computations, optimisation, networking protocols, data communication and decision analysis, control theory, automation, signal processing, mechanical design, CAD/CAM and simulation systems. In 2006, he has changed his last name from ‘Mustafa’ to ‘Dalalah’.

Copyright © 2013 Inderscience Enterprises Ltd.
1 Introduction

Road transport accounts for a relatively high percentage of all goods traffic as well as passenger traffic. Particularly, the road portion of the goods traffic has been increasing constantly which may hit almost 47% by the end 2010 (Chow, AHF, 2007; European Commission, 2001).

Intense road traffic provokes problems of pollution, road congestion, and traffic jam and incident supervision in every metropolis. Reported by the European Commission (2001), in Europe transport was responsible for 28% of carbon dioxide emissions, of which road transport alone accounts for 84%. Heavy road traffic also raises travel delays and congestion. It has become usual to observe vehicles crawling slowly along busy streets in urban areas. Moreover, heavy traffic may result in complicated management of unexpected events. Such events, together with the heavy daily traffic, induce heavy congestion and therefore delays for thousands of travellers (Cheng et al., 2005).

Due to poor transportation infrastructure in most of the developed countries which is originally designed to carry a definite amount of traffic, such infrastructure is often congested by a devastating demand of resources: this is the case of urban streets, railroads, airplane connections (Scellato et al., 2010). Managing the ever increasing amounts of road traffic is vital for logistics and economy. However, expanding and improving existing road networks are often restricted by increasingly tight fiscal and other constraints such as the physical and environmental issues (Hau, 1998). Had known these constraints, transport planners and engineers have to design and implement effective strategies to manage the transport facilities and infrastructure and to accommodate the new traffic demands. To achieve this, a reliable traffic flow control/assignment should be implemented by which the travellers will respond to traffic management measures and evaluation so as to manage the heavy traffic on congested roads (Murchland, 1970; Kelly, 1991).

The traffic assignment can be defined as the process of determining the flow pattern in a network with a specified travel demand between the pairs of origin-destination. Optimal assignment finds the flow patterns in order to put the total travel time cost in the network to a minimum value. Traffic assignment has long been a vital step in the planning of transportation (Jayakrishnan et al., 1994).

On the other hand, a congestion pricing or congestion charge is a system of surcharging users of a transport network in periods of peak demand to reduce traffic congestion. Studying congestion pricing is essential to ease traffic and promote transit. In urban traffic networks, the theory of marginal cost pricing has been extensively used as the major technique for congestion pricing. It is well known in the literature that by commanding the congestion tolls to be precisely equal to the difference between marginal societal cost and marginal private cost, a user equilibrium flow model can be advanced toward a system optimal flow model in an optimal situation of a system.
The literature classification of congestion pricing depends on the analysis used. Some literature studies handle the transportation problem as a static environment (Lou et al., 2010), and the others are based on a dynamic analysis (Bie and Lo, 2010). The static congestion pricing models entails a sole bottleneck or a broad traffic network at steady-state condition at all times and therefore the travel demands and costs patterns are not dynamic over time. Such stationary models do not take into consideration the temporal effect of delays or cost terms at the current time on potential changes in the intensity of traffic jamming. An efficient taxation system can be estimated via the theory of marginal cost pricing, Walters (1961). By equating the marginal private costs to marginal social costs Walters could present a formal solution for efficient taxes and suggested a mixture of mileage taxes, gasoline taxes and congestion tolls. Developed by Dafermos and Sparrow (1971), a static congestion pricing model was also used to determine a system of optimal tolls on general traffic networks.

The link-toll and path-toll collection problems as well as the toll pattern for multiclass-user transportation networks were addressed by Dafermos (1973) using the marginal cost pricing theory. It was found that when the demand and cost functions satisfy certain smoothness conditions, the marginal cost tax is optimal. The results of Dafermos were also generalised to the case where the demand and cost functions are non-separable (Smith, 1979).

Another category of congestion pricing models is dynamic, here, the pattern of travel demands and costs differ over time and accordingly the related congestion tolls need to be time-varying as well (Wie and Tobin, 1998). For instance, the importance of departure time decisions and scheduled delays were considered by Henderson (1974) in modelling of congestion pricing for a single bottleneck. His results showed that the commuter’s decision of departure time is affected by the time-varying congestion tolls which give rise to a competent reshuffle of traffic flows as compared to a non-toll condition.

A user preferred flow pattern could be shifted toward a socially preferred one as presented (Carey and Srinivasan, 1993). The author derived marginal private costs, marginal social costs, and congestion externality costs and established a system of optimal congestion tolls to achieve such goal. A dynamic congestion pricing model of a congested network was formulated by Huang and Yang (1996) for parallel routes with elastic travel demand which implements the optimal control theory.

The dynamic assignment models proposed by Merchant and Nemhauser (1978), Carey (1987), Friesz et al. (1989), Wie (1988), and Wie et al. (1990, 1994) absolutely assume that adjustments in density spread instantly across the arc. This short-coming turns to be significant when vehicles come into an empty arc. While these vehicles have not traversed the arc, the rate of depart flow immediately becomes positive. Alternatively, in our model, each arc is assumed to consist of a congested segment with a bottleneck where queues are formed and uncongested segment with a constant free-flow travel time.

Traffic optimisation has always been a critical subject in the perspective of communication and transportation systems (Wardrop and Whitehead, 1952; Helbing, 2001; Chowdhury et al., 2000; Pastor-Satorras and Vespignani, 2004; Belomestny et al., 2003). The study of congestion in complex networks has mainly focused on information systems, such as grid computing networks or flow optimisation over the internet (Pastor-Satorras and Vespignani, 2004; Boccaletti el al., 2006). Diverse solutions have been proposed in order to increase the network load and to avoid traffic jams (Guimer et al., 2002). Most of such schemes considered the dynamic routing to the less congested paths which had a significant impact on improving the network capacity (Echenique
et al., 2004, 2005). Some literature showed that using the network limited and local knowledge can largely improve routing and ability to better navigation (Tadic and Rogers, 2002; Kujawski et al., 2006). Others went further by implementing a cellular automata model on a complex network to simulate the motion of vehicles along streets, coupled with a congestion aware routing at street crossings (Scellato et al., 2010).

The primary objective of this paper is to develop a model under the assumptions that arc capacities are stable and commuters have the aptitude to gain knowledge of the best route choices ultimately through day-to-day travelling and observing the delays and surcharges. The model should react interactively to reach a dynamic system optimal flow pattern. The optimal flow pattern can be enforced by simply introducing a pricing system that is extracted from some function of the optimal flow rates. According to the entailed charges, the habits of the network users can be driven to a situation that reflects the optimal traffic rates. In the presented model, the optimal traffic rates will maximise the social benefits subject to network tolerable operational costs (i.e., private costs).

Our model is characterised by, first, the imposition that congestion affects the commuter’s decision of route and hence, the traffic flow on that route, by modifying their departure time, travel mode, destination, vehicle occupancy, or trip frequency. Second, we assume a first-in-first-out queue discipline, a convenient assumption; however, in modelling vehicle dynamics, this may not be practical. Third, the network surcharges affect the travellers’ behaviour, where the surcharges can be set to reflect the optimal rates.

The presented model provides a measure of the social system benefit which may be considered as qualitative measure of the level of service attributes, time, comfort of the users subject to network tolerable costs paid for the operation, congestion management, air pollution, noise, accidents, maintenance, etc. Hence, the model tends to maximise the social benefit under the network willingness to pay for provided the transportation service. It was noticed that toll and travel costs, although may affect the commuters decisions and the travel conventions, such as changing the routes, time or mode of travel, however, travel habits are always restricted with the available transportation configuration, topology and infrastructure, hence, it is more realistic to accommodate the traffic flow to the existing situation rather than introducing infeasible solutions to the commuters as well as the network. In our model, we address this matter by restricting the traffic flow rates to the maximum level of service a network is willing to afford.

The paper is organised according to the following: Section 2 explains the transportation system and puts the model in a non-linear programming problem. In Section 3, we explain the solution of the non-linear model. Next in Section 4, a comparison between the numerical and analytical solutions is presented. The breakeven analysis between the network social and private costs is illustrated in Section 5 while Section 6 presents further simulation experiment to demonstrate the rate-to-queue proportionality. Finally, we present the conclusions in Section 7.

2 The transportation model

In this section, we propose a non-linear transportation system network model that depends on cost pricing in urban traffic networks. Here, we implement the idea that a user equilibrium flow pattern can be shifted toward a system optimal flow pattern by imposing upper bounds on the traffic rates and the network private costs. Here, we expect
that the imposition of queuing delays can encourage commuters to act in a desired manner so as to maximise the overall benefits subject to some network tolerable cost upper bounds.

Consider a transportation network modelled by a directed graph $G(A, N)$, where $A$ is the set of $m$ directed arcs (i.e., roads) and $N$ is the group of nodes (i.e., conjunctions). Suppose further that the arcs have deterministic transportation load (i.e., capacities) given by $L_j$ (vehicle/unit time) $\forall j \in A$. A set $R$ consisting of $n$ different routes will pass through the transportation network where each route type $i \forall i \in R$ has its own path that traverses a subset of the existing arcs.

Suppose now the minimum time required traversing route $i$ in the case of uncongested scenario and predetermined speed limits is $t_i$, which is hypothetically a known quantity at the traffic sources. Denote, further, the traffic rates of the set of existing routes by $x = \{x_1, x_2, \ldots, x_n\}$ and the whole time to complete a route including the delays by $T_i(x)$, which is basically a function of the traffic rate of route $i$ and the other crossing traffic coming from other routes.

Here, we introduce an evaluation of the system benefits by the summation of the function $B_i(x_i)$ $\forall i \in R$, where the sum of the system benefits is a function of the traffic rates. The function $B_i(x_i)$ is separable, differentiable, concave, additive and strictly increasing by assumption. It may well stand for an evaluation of the returns or utility from using the transportation network. Such benefit functions states that more returns are attained upon increasing the flow rate, however, this increase in the returns is marginal for high flows. Besides, Each route $i$ traverses a subset of arcs $A(i) \subseteq A$, whereas each arc $j$ will comprise a subgroup of routes $R(j)$ that will pass through the arc $j$, i.e., $R(j) \subseteq R$. Accordingly, the total system benefits can be computed by the expression $\sum_{i \in R} B_i(x_i)$.

By assumption, each arc consists of an uncongested segment with a constant free-flow travel time and a congested segment with a bottleneck where traffic queues builds up. A private network cost term ($c_i$) will be associated with each route, where the network spends $c_i$ for the operation of route $i$. This cost component can be regarded, for instance, to the operation and maintenance of the related route and any other miscellaneous costs paid by the network. This cost component is expressed in terms of monetary value per vehicle per unit time (e.g., $$/veh/hr).

Had such notation set up, the optimal rates of such system will solve the subsequent optimisation problem:

$$\begin{align*}
\max \sum_{i \in R} B_i(x_i) \\
\text{subject to :} \\
x_i (T(x) - t_i) c_i \leq C_i \max \forall i \in R \\
0 \leq x_i \leq U_i
\end{align*}$$

(1)

where

- $B_i(x_i)$ The anticipated benefit function of using route $i$ at a rate $x_i$.
- $x_i$ The traffic rate across route $i$, (veh/unit time).
- $T(x)$ The total time required to complete route $i$. 

The minimum time required to complete route $i$.

$c_i$ The network private cost for the operation of route $i$, ($$/veh/unit time$).

$C_i^{\text{max}}$ The upper bound on the network private costs, ($$/unit time$). It represents the maximum the network is willing to pay for the operation of route $i$.

$U_i$ The upper limit on the traffic rate of route $i$.

The rate vector $x$ represents the decision variable of the above optimisation problem under the given constraints. The optimisation problem will maximise the global benefits of the network users over a bounded feasible solution space.

In our study, we also present the concept of rate-to-cost proportionality which will be explained shortly in subsequent sections. For this reason, we prove that the left expression of the first constraint is an approximation of the cost of traffic in transit. In essence, this is valid provided that the quantity of vehicles belonging to route $i$ in some queue $j$ is proportional to the fraction $x_i / L_j$, which is a fact that has been fundamentally recognised in queuing theories. For each route $i$, the whole time exhausted in the network $T_i(x)$ is equivalent to the summation of the queuing delay $\phi_j$ and the minimum time $t_i$.

More specifically, that is:

$$T_i(x) = t_i^i(x) + t_i \quad \forall i \in \mathbb{R}$$

(2)

At the moment, for any arc $j$, let $q_j$ indicate the noticed queue size and define the queuing delay $\phi_j$ as the time required to empty the observed queue. In that case, the queuing delay at arc $j$ is measured by:

$$\phi_j = \frac{q_j}{L_j}$$

(3)

Using the proportionality assumption stated above, the number of vehicles belonging to route $i$ at arc $j$ is given by $q_jx_i / L_j$. With the aid of the equation (3), we may easily show that $q_jx_i / L_j = x_i\phi_j$, where the entire number of vehicles in queue that belong to route $i$ is therefore given by:

$$\sum_{j \in A(i)} x_j\phi_j = x_i \sum_{j \in A(i)} \phi_j$$

(4)

Clearly, the summation term inside the (RHS) of the above expression is the same to the collective queuing delay of route $i$ (i.e., $t_i^i(x) = \sum_{j \in A(i)} \phi_j$). The traffic waiting in queues of route $i$ is thus estimated by $x_i t_i^i(x)$ while that in transit can be estimated by $t_i x_i$. Thus, the (LHS) of the first constraint is an approximation of the total private cost paid by the network per unit time on route $i$, where $C_i^{\text{max}}$ is the upper bound the network is willing to pay.

The first constraint in (1) is non-linear and dynamic, which means it changes over time. Due to its dynamic nature, the entire delay $T_i(x)$ of any route will be computed via real-time monitoring where the solution is synchronised to time-based iterations. As a result, $T_i(x)$ is updated frequently in a timely manner to assist in presenting the numerical solution of the optimisation problem as will be explained later in this paper.
As for the second constraint, the model entails an upper and lower bounds on the traffic rates of each used route in problem (1). The upper and lower bounds should have positive values.

3 Dynamic system optimal assignment

For the case of relatively heavy traffic traversing the transportation network, we assume that the cross traffic of other flows has the major effect on the delay on route \( i \) at a certain conjunction (Kelly et al., 2003; Dalalah, 2010; Dalalah and Araidah, 2010; Dalalah, 2008). As a result, for a busy bottleneck road with numerous inward routes, the effect of route \( i \) on the road total delay is understood to be insignificant and may be ignored while more routes bypass all the way through the bottleneck road. This entails that the total delay at an arc is primarily owing to all inward traffic from the different routes; however, the effect of the rate of any specific route on the whole delay is insignificant. This can be expressed as:

\[
\frac{\partial T_i(x)}{\partial x_i} \approx 0 \quad \forall i \in \mathbb{R}
\]

(5)

For the first constraint set, Lagrange multipliers vector \( \lambda \) is addressed to help find the probable KKT points that optimises the related objective function. In fact, there is an optimal solution of the rates for each value of \( \lambda \), yet, the solution is not essentially the inclusive optimal. Among the different values of \( \lambda \) there will be a vector \( \lambda^* \) with a unique rate vector \( x^* \) that will maximise the problem in (1).

3.1 The model dual problem

In this section, we address the optimisation problem by means of the duality analysis presented in Bertsekas (1997). Thus, the Lagrangian of problem (1) can be given by the following equation:

\[
L(x, \lambda) = \sum_{i \in \mathbb{R}} B_i(x_i) - \sum_{i \in \mathbb{R}} \lambda_i \bigg( x_i c_i (T_i(x) - t_i) - C_i^{\text{max}} \bigg)
\]

\[
= \sum_{i \in \mathbb{R}} \left[ B_i(x_i) - \lambda_i \left( x_i c_i (T_i(x) - t_i) - C_i^{\text{max}} \right) \right]
\]

(6)

where the vector of Lagrange multipliers is given by \( \lambda \). Consequently, the dual problem of (1) gives the following objective function and constraints:

\[
D(\lambda) = \max_{\theta \in \mathcal{U}} L(x, \lambda)
\]

(7)

where \( \mathcal{U} \) is the set of routes’ upper limits on the traffic rates. The dual problem states that there is a dual vector \( \lambda = (\lambda_i, \ i \in \mathbb{R}) \) that should be chosen to \( \min_{\lambda \geq 0} D(\lambda) \) which is equivalent to:

\[
\min_{\lambda \geq 0} \left( \max_{\theta \in \mathcal{U}} L(x, \lambda) \right)
\]

(8)
First, let us address the maximisation of the expression in brackets provided the constraint $0 \leq x \leq U$, this is:

$$\max \sum_{i \in \mathbb{R}} B_i(x_i) - \sum_{i \in \mathbb{R}} \lambda_i \left( x_i c_i (T_i(x) - t_i) - C_i^{\text{max}} \right)$$

s.t.

$$0 \leq x_i \leq U_i$$

(9)

Problem (9) has the following Lagrangian:

$$l(x, \alpha, \beta) = \sum_{i \in \mathbb{R}} B_i(x_i) \sum_{i \in \mathbb{R}} \lambda_i \left( x_i c_i (T_i(x) - t_i) - C_i^{\text{max}} \right) + \sum_{i \in \mathbb{R}} \alpha_i (U_i - x_i) + \sum_{i \in \mathbb{R}} \beta_i x_i$$

(10)

where $\alpha_i$ and $\beta_i$ are the two bounds’ multipliers of the constraint in problem (9). Bring to mind that $\partial T_i(x) / \partial x_i \approx 0$ from (5), subsequently, the KKT optimality conditions are respectively given by:

$$B_i(x_i) - \lambda_i c_i (T_i(x) - t_i) - \alpha_i + \beta_i = 0$$

$$\alpha_i \geq 0$$

$$\beta_i \geq 0$$

$$\alpha_i (x_i - U_i) = 0$$

$$\beta_i x_i = 0$$

∀ $i \in \mathbb{R}$

Notice that, for the fourth expression above, if $\alpha_i > 0$ then $x_i = U_i$ as an obligation for the expression to hold. Alternatively, if $\alpha_i = 0$, the first term has to hold as the 4th expression, if binding, will not maximise the objective function for separable problem of this type provided that the separable objective function components in this case are all strictly increasing. Correspondingly, for the very last term, either $\beta_i$ equals to zero or $x_i = 0$. When $\beta_i = 0$, $x_i$ can be found using the first expression. However, if $x_i = 0$, this solution set will not maximise a strictly increasing separable sum of objective functions.

In conclusion and supposing that the inverse of $B_i'$ exists, the collection of the probable KKT points will be:

$$x_i (\hat{\lambda}_i) = 0$$

$$x_i (\hat{\lambda}_i) = U_i$$

$$x_i (\hat{\lambda}_i) = B_i^{-1} \left( \hat{\lambda}_i c_i (T_i(x) - t_i) \right)$$

∀ $i \in \mathbb{R}$

The optimal solution can be a set of any of the above expressions. Although at definite $\hat{\lambda}$ values the solution can be a mixture of the above, yet we will show in the next paragraph that for such objective function, only one candidate will maximise our problem in (9).

Note that for any $i \in \mathbb{R}$ whenever $x_i$ equals to zero, the lower bound of that rate will be active for that particular route. However, as the objective function in (1) has separable and strictly increasing constituents, this tells that a lower bound of ‘0’ will not maximise the objective function. Further delay investigation is needed for the two remaining possibilities which will also help decentralise the result. Generally, the 2nd and 3rd
solution candidates are functions of the delay, when no queuing delay is experienced (i.e., $T_i(x) = t_i$) the objective function of (9) will be firmly rising, which means that $U_i$ is the sole probable solution. In contradiction to this, if queuing delay is experienced, the private cost constraint will be in command of harmonising the traffic rates. Consequently, we may summarise the resolution of problem (9) as a function of $\lambda$ by:

$$x^*_i = \begin{cases} x_i(\lambda_i) = U_i & \text{if } T_i(x) = t_i \\ x_i(\lambda_i) = B^{-1}_i(\lambda_i c_i(T_i(x) - t_i)) & \text{otherwise} \end{cases} \quad \forall i \in \mathbb{R}$$ (11)

Note that we can shove the solution to reside on an interior spot of the solution space in (9) by raising $U_i$ to a considerably big number. Accordingly, the gradients of the dual problem (8) in relation to $\lambda$ are expressed as:

$$\frac{\partial L(x, \lambda_i)}{\partial \lambda_i} = -\{x_i(c_i(T_i(x) - t_i)) - C^\text{max}_i \} \quad \forall i \in \mathbb{R}$$ (12)

By means of the gradient projection algorithm, an estimation of $\lambda$ can be acquired at the route sources by minimising the dual objective function with respect to the multiplier $\lambda$. Following sufficiently small step sizes, the gradient projection algorithm could be verified to converge. Thus, $\lambda$ can be computed in an iterative manner as follows:

$$\lambda^{k+1}_i = \left[ \lambda^{k}_i + \gamma \left( x_i c_i(T_i(x) - t_i) - C^\text{max}_i \right) \right]^+$$ (13)

where the step size is given by $\gamma$, the superscript $k$ is the iterations index and $[y]^+ = \max\{0, y\}$. In our algorithm and by pre-assumed network setup, the quantity $T_i(x)$ will be conveyed from the destination directly back to the source of each route at which the queuing delays can be estimated. The quantity $T_i(x)$ is simply estimated by the time between the departure and arrival of the vehicles of each route. The concluding algorithmic clarification of the optimisation problem will accordingly look like:

$$x^*_i = \begin{cases} x_i(\lambda_i) = U_i & \text{if } T_i(x) = t_i \\ x_i(\lambda_i) = B^{-1}_i(\lambda_i c_i(T_i(x) - t_i)) & \text{otherwise} \end{cases} \quad \forall i \in \mathbb{R}$$ (14)

The expression above conforms to the law of demand and supply. If the demand $x_i(t^*_i(x)c_i)$ to a number of vehicles goes beyond the supply $C^\text{max}_i$ the price $\lambda_i$ will be shifted up, otherwise shifted down. With this solution, the arcs of the transportation network can be understood as distributed processors that solve the dual problem in a computational system. At any iteration, the traffic sources separately solve (14) and converse their results to the roads that will generate new charges estimated by $\lambda, c_i(T_i(x) - t_i)$ which finally will be experienced by the traffic sources.

If for a route, there is no queuing delay encountered, then $T_i(x) = t_i$ or alternatively, $t^*_i(x) = 0$ and the cost constraint will not be significant anymore, rather, the upper bound on the rates should be the only binding limit for such objective function. As a result, for a zero delay scenario, we have $= B^{-1}_i(\lambda c_i(T(x) - t_i)) = B^{-1}_i(0) = U_i$. 


For the case when \( T_i(x) > t_i \), some queue delay is expected along the route \( i \). In such scenario, the lower bounds ‘0’ are unlikely to maximise a strictly increasing objective function, rather, the cost constraint will be binding where \( x_i(\lambda) = B_i^{-1}(\lambda, C_i(T(x) - t_i)) \). Further, as the computed rates should move toward the lower bound as elevated charges (i.e., queueing delays) are noticed, this can be understood as \( B_i^{-1}(\infty) = 0 \), or else infeasible rates can be found if \( B_i^{-1}(\infty) < 0 \). Similarly, if \( B_i^{-1}(\infty) > 0 \), underestimation of the charges may come across.

It is noteworthy to mention that any previously defined benefit function will define \( B_i^{-1} \) and vice versa. Further, any function \( B_i^{-1} \) that match with the listed boundary conditions above should be satisfactory. The conditions stated in Section 2 along with the above two boundary conditions of \( B_i^{-1} \) can describe diverse collection of benefit functions. Here, we present a suggested benefit function and its corresponding \( B_i^{-1} \) in Table 1.

<table>
<thead>
<tr>
<th>Benefit function ( B(x) )</th>
<th>Charge function ( B_i^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{x_i}{\nu_i} \left( 1 - \log \left( \frac{x_i}{U_i} \right) \right) )</td>
<td>( U_i e^{-\left{ \nu_i C_i (T_i(x) - t_i) \right}} )</td>
</tr>
</tbody>
</table>

Table 1 lists a new factor \( \nu_i \) which is a shape multiplier that controls the concavity of \( B_i^{-1} \). It notifies how ravenous a route traffic to a resource (road). Dissimilar values of \( \nu_i \) may cause a traffic type to be shoved into the network quicker than the others yielding more fluctuations as will be illustrated abruptly. Ultimately, all route types will have to concur on the optimal allocation.

### 4 Analytical vs. algorithmic solutions

Our solution algorithm can perform in a decentralised manner while the traffic sources do not communicate with the network or with each other. We summarise the algorithmic steps as follows:

1. The traffic sources will update the multiplier \( \lambda_i \) as follows:
   \[
   \lambda_i^{k+1} = \left[ \lambda_i^k + \gamma \left( x_i C_i (T_i(x) - t_i) - C_i^{\text{max}} \right) \right]^+ 
   \]
   where \([y]^+ = \max\{0, y\}\), \(k\) is an integer iterations index, \(\gamma\) is a small step size and \(\lambda_{i,\text{initial}} = 0\).

2. The rates at the traffic sources will also be updated as follows:
   \[
   x_i(\lambda_i) = B_i^{-1} \left\{ \lambda_i C_i (T_i(x) - t_i) \right\} 
   \]
Obviously, the above steps can be put into practice in a distributed way and can guide the transportation network to maximise its global benefit listed in (1). The sources will have to evaluate step one and two in real-time operation to be able to estimate $T(x)$, meaning that, this quantity has to be estimated after each step in the real operation or simulation of the network.

To have more insights to the problem, consider a simple example consisting of one road (i.e., one route) with a capacity of 120 veh/min and network private cost of $c_i = \$0.5$/veh/min, Figure 1. The upper limit on the total private cost is $\$5$/min and that for the traffic rates is $U_1 = 200$. If $U_1 < 120$ then the traffic rate will remain constant ($= U_1$) since no delay will be observed. Now, since the upper bound overshoots the road capacity, it is expected to observe some delay in the route.

**Figure 1** Single road, single route problem

We simulate the above example using ARENA 10 following the given steps above where $\gamma = 1$, $\nu_1 = 0.1$. Figure 2(a) and Figure 2(b) show that once the rate is little higher than the road capacity, the cost constraint will be in charge of reducing the traffic rate to reach the predetermined capacity (120 veh/min) even it is anonymous to the sources and destinations. The Lagrange multiplier will also converge to a certain value that will maximise the overall utility. Although the rate in this solution has to begin with the upper limit, it will reduce quickly to move toward the maximum road capacity since queues will start to build up and more delay is observed which will move the optimal point to some rate less than $U_i$ as seen in Figure 2. Further, altering the algorithm parameters will yield in dissimilar performance, however, the optimal solution does not change. It was noticed that the smaller the step-size $\gamma$ the longer the time needed to achieve optimality. Increasing the factor $\nu$ may raise the fluctuations and amplify the peaks, while increasing $\gamma$ will extend the peaks over a wider period. For example, if we put the parameter $\nu$ equal to 10, we see that the rate gets greedier resulting in further oscillations as depicted in Figure 3.

**Figure 2** Simulation results, $\nu = 0.1$, (a) the search for the optimal rate (b) the optimal Lagrange multiplier (see online version for colours)
Considering the analytical solution, recall that there was some queuing delay due to the oscillation in Figures 2 and 3, which means that sources were delivering at the maximum capacity, i.e., $x_1 = 120$ veh/min, or in other words, $x_1 t_1^Q(x)c_1 - C_{1}^{\text{max}} = 0$ and $t_1^Q(x) = C_{1}^{\text{max}} / (x_1 c_1) = 5 / (120 \times 0.5) = 0.08333$ min. Plugging the above values into $B_{i-1}^i$ function we get:

$$x_1 = U_i e^{-\left[\eta + \lambda_i \left(\frac{1}{t_1^Q(x)}\right)\right]} = 200e^{-[0.1x + 0.5(0.08333)]} = 120$$

Solving the above equity will result in $\lambda_1 = 122.598155$, which is the same result obtained via simulation using the presented algorithm. This example shows that, following the given traffic rate pattern will yield a private cost of $\$5$/min, that is, $x_1 t_1^Q(x)c_1 = \$5/hour$. Note further, that the cost component is 100% associated with the users of route 1 where ten vehicles are expected to be in queue. In case of other routes sharing the same bottleneck, the cost components will be divided according to the cost proportions as will be illustrated in the simulation part. Note hat since the private cost should be less than the social cost, then the social cost should be $\geq 5 / x_1$, meaning that the network should charge the users an amount larger than $0.0417/\text{veh}$.

## 5 Charges breakeven analysis

One of the main assumptions in the introduction part of this study was that the commuters travel habits may well change according to the surcharges imposed by the network. To this end, our model can calculate the optimal traffic rates that will maximise the social benefits of the commuters subject to some network predetermined operational costs. For very large rate upper bounds ($U_i$), the network private cost constraint will be binding, at which the network private costs are estimated by the following formula:

$$\text{Per-route network private cost, } p_i^j = c_i \sum_{j \in A(i)} q_j^i$$  \hspace{1cm} (15)
where \( q_j^i \) refers to the vehicles in queue at arc \( j \) that belong to route \( i \). Equation (15) gives the cost paid by the network to compensate for the operation of the routes. The total network private cost is simply calculated by \( \sum_{i \in R} p_i^j \). Meanwhile, the system optimal rates are calculated using the presented model for each route. Now following the assumption made in the beginning of this section by which the travel habits can be affected by the toll charges, we can compute the commuters’ breakeven charges (social costs) of each route by simply dividing the per-route network private cost by the corresponding route optimal rate, particularly:

\[
h_{\text{breakeven}} = \frac{c_i \sum_{j \in A(i)} q_j^i}{x_j}
\]

where \( h_{\text{breakeven}} \) (\$/veh) is the network breakeven charges to be tolled from the commuters. This amount is needed to breakeven the network operational costs (i.e., \( \sum_{i \in R} C_i^{\text{max}} \)).

Essentially, for a profitable operation of the network, the social costs have to be higher than the private costs, where the equality between both represents the breakeven point. Hypothetically, a successful operation of the network requires that the profit margins maintain the proportionality between the different routes with respect to their breakeven points under the assumption that travel attitudes can be affected with the magnitude of charges.

For further demonstration, consider the transportation network having the configuration given in Figure 4. Here, three different routes cross the network using the given arcs (roads). The capacity of the two roads is 100 veh/min each and the network private cost is $0.5/veh/min for all the routes. The upper bound of the network tolerable private cost is set to $5 for the three routes. After conduction a simulation run using the presented steps in Section 4 the solution best rates were: \( x_1 = 33.333 \) veh/min, \( x_2 = 66.667 \) veh/min and \( x_3 = 66.667 \) veh/min. At the steady state, the two queues \( q_1 \) and \( q_2 \) will have a total of 15 veh each with the corresponding ratios: \( q_1^1 / q_2^1 = 1/2 \) and \( q_1^2 / q_2^2 = 1/2 \). The proportionality assumptions made in Section 2 are shown to be convincing from the simulation experimental results given in Table 2 where the calculated optimal rates conform to this proposition, here, \( x_1 / x_2 = 1/2 \) and \( x_1 / x_3 = 1/2 \).

Further, we may notice that \( \sum_{j \in A(i)} q_j^i = C_i^{\text{max}} / c_i, \forall i \in R \), however, this will hold under the assumption of significantly large \( U_i \) values.

**Figure 4** Two arcs and three routes transportation network
Table 2  The solution of the configuration in Figure 4

<table>
<thead>
<tr>
<th>Route (i)</th>
<th>Rate (veh/min)</th>
<th>$C_i^\text{max}$ (S/min)</th>
<th>$c_i$ (S/veh/min)</th>
<th>$C_i^\text{max} / c_i$ (veh)</th>
<th>$q^*_i$ (veh)</th>
<th>$q^*_i$ (veh)</th>
<th>$\sum_{[j_0,i_0]} q^*_i$ (veh)</th>
<th>Cost, $p^*_i$ (S/min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>33.333</td>
<td>5</td>
<td>0.5</td>
<td>10</td>
<td>5</td>
<td>5</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>66.667</td>
<td>5</td>
<td>0.5</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>66.667</td>
<td>5</td>
<td>0.5</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>Network total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>15</td>
<td>15</td>
<td>30</td>
<td>$15$</td>
</tr>
</tbody>
</table>

Note: The steady state rates and queues are presented.

For this particularly simplified example the breakeven charges of the above optimal rates can be calculated using equation (15). The charges are given in Table 3, where we can spot the highest charge that belongs to route 1 as it traverses two links of the given network. We would like to bring the attention that the charges are relatively small due to the small road capacities and low tolerable operational costs given that they represent the breakeven points and not necessarily what the network may have to charge.

Table 3  Network breakeven per-route charges

<table>
<thead>
<tr>
<th>Route</th>
<th>Breakeven charges, $h_{\text{breakeven}}$ (S/veh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.15</td>
</tr>
<tr>
<td>2</td>
<td>0.075</td>
</tr>
<tr>
<td>3</td>
<td>0.075</td>
</tr>
</tbody>
</table>

6  Beyond the simulation results

Several simulation experiments of the algorithm will be presented in this section. We use ARENA discrete event simulation package to illustrate the idea. Here, we would like to draw the attention that all the routes have identical benefit functions through all the simulation runs. Further, the new concept of rate-to-queue proportionality will be demonstrated through the experiments for better understanding of the idea.

Consider the last example given in the last section and alternatively assume the following per-route private costs: $c_1 = \$0.5$, $c_2 = \$0.4$, $c_3 = \$0.1$ per veh/min and the maximum tolerable bounds for the operation of the routes: $C_1^\text{max} = \$5$, $C_2^\text{max} = \$3$, $C_3^\text{max} = \$2$ per minute with the same given road capacities. After conducting a simulation run on this scenario we found the following results as presented in Table 4.
Cost driven traffic assignment in transportation networks

Table 4 Rates and queues at the steady state

<table>
<thead>
<tr>
<th>Route (i)</th>
<th>Rate (veh/min)</th>
<th>$c_i$ ($/min)</th>
<th>$C_i^{\text{max}}$ ($/veh/min)$</th>
<th>$C_i^{\text{max}} / c_i$ (veh)</th>
<th>$q_i^1$ (veh)</th>
<th>$q_i^2$ (veh)</th>
<th>$\sum_{j \in A(i)} q_j^i$ (veh)</th>
<th>Cost, $p_i^c$ ($$/min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>26.6667</td>
<td>5</td>
<td>0.5</td>
<td>10</td>
<td>2.72727</td>
<td>7.27273</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>73.3333</td>
<td>3</td>
<td>0.4</td>
<td>7.5</td>
<td>7.5</td>
<td>0</td>
<td>7.5</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>73.3333</td>
<td>2</td>
<td>0.1</td>
<td>20</td>
<td>0</td>
<td>20</td>
<td>20</td>
<td>2</td>
</tr>
</tbody>
</table>

Network totals 10.22727 27.27273 37.5 $10

The presented maximisation model has some remarkable properties to the extent that it is possible to calculate the optimal rate using exact analytical solution. For instance, since route 2 has a single bottleneck, then a total of $C_2^{\text{max}} / c_2$ vehicles will be queued on its queuing segment, this yields $q_2^1 = 7.5$, while $q_2^2 = 0$ since the route simply does not go to arc 2. Similarly, for route 3, we have $q_3^1 = 0$, while $q_3^2 = 20$, and for route 1, we know already that a total of 10 vehicles will be queued along this route, that is $q_1^1 + q_1^2 = C_1^{\text{max}} / c_1 = 10$. To find the optimal rates now, we simply hold the proportionality between the routes at any conjunction, i.e., $x_1 / x_2 = q_1^1 / q_2^1$ and $x_1 / x_3 = q_1^1 / q_3^1$ for this scenario. Recall further that under the steady state we have $x_1 + x_2 = 100$ veh and $x_1 + x_3 = 100$ veh. To this end we have constructed five equations with five variables, particularly, $x_1, x_2, x_3, q_1^1, q_1^2$. Solving the given equations will result in the following rates: $x_1 = 26.6667, x_2 = 73.3333, x_3 = 73.3333, q_1^1 = 2.7272, q_1^2 = 7.2727$. Clearly, the simulated and analytical solutions are the same where the rates are proportional to their queues. The breakeven charges are also given in Table 5. Once again, the charges are noticed to be relatively low due to the small road capacities and cost upper bounds.

Table 5 Network breakeven charges of Table 4

<table>
<thead>
<tr>
<th>Route</th>
<th>Breakeven charges, $h_{\text{breakeven}}$ ($$/veh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1875</td>
</tr>
<tr>
<td>2</td>
<td>0.0409</td>
</tr>
<tr>
<td>3</td>
<td>0.0273</td>
</tr>
</tbody>
</table>
Table 6
The results of the configuration in Figure 5, i.e., the steady state rates and queues

<table>
<thead>
<tr>
<th>Route (i)</th>
<th>Rate (veh/min)</th>
<th>$c_i$ ($/min)</th>
<th>$c_i$ ($/veh/min)</th>
<th>$C_{i}^{max}$ (veh/min)</th>
<th>$q_i$ (veh)</th>
<th>$q_i$ (veh)</th>
<th>$q_i$ (veh)</th>
<th>$q_i$ (veh)</th>
<th>Total $q_i$ (veh)</th>
<th>Cost, $p_i$ ($/min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>21.227297</td>
<td>6</td>
<td>0.5</td>
<td>12</td>
<td>0</td>
<td>12</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>46.227297</td>
<td>3</td>
<td>0.5</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>28.772703</td>
<td>2</td>
<td>0.1</td>
<td>20</td>
<td>0</td>
<td>16.2654919</td>
<td>0</td>
<td>3.7345081</td>
<td>20</td>
<td>2</td>
</tr>
<tr>
<td>Network totals</td>
<td></td>
<td>28.2654919</td>
<td>9.7345081</td>
<td>38</td>
<td>$11$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

D. Dalalah and W.A. Al-Rawabdeh
In Figure 5, we show different transportation system pattern with one benefit function for all routes. The road capacities are $L_1 = 100\,\text{veh/min}$, $L_2 = 50\,\text{veh/min}$, $L_3 = 150\,\text{veh/min}$ while the private costs are $c_1 = 0.5\,\text{$/veh/min}$, $c_2 = 0.5\,\text{$/veh/min}$, $c_3 = 0.1\,\text{$/veh/min}$ with upper bounds ($C_i^{\text{max}}$) of $6\,\text{$/min}$, $3\,\text{$/min}$, $2\,\text{$/min}$, respectively.

**Figure 5** Four arcs and three routes transportation system

Table 6 presents the simulation output of such system. All over again, the simulation experimental results conform to the proportionality proposition where the rates and queues are proportional. For illustration, we may effortlessly validate that $x_1 / x_3 = q_1^1 / q_3^1$ and $x_2 / x_3 = q_2^1 / q_3^1$. Moreover, the entire number of vehicles in queue of every route is equal to $C_i^{\text{max}} / c_i$.

Similar analytical solution can be found after identifying the bottlenecks of the above system. The simulation output ensures the fact that the selection of the preferred level $C_i^{\text{max}}$ directly affects the optimal solution. For the above scenario, we will have a total of five variables, namely, $x_1, x_2, x_3, q_2^1, q_3^1$ with five relate equations, among which, two for the bottlenecks: $x_1 + x_3 = 50$, $x_2 + x_3 = 75$, two for the proportionality: $x_1 / x_3 = q_1^1 / q_3^1$, $x_2 / x_3 = q_2^1 / q_3^1$ and one for the overall sum of queues in route 3, that is $q_3^1 / q_3^1 = 20$. Solving this system of equations will yield the same simulation results above.

The network breakeven charges are listed in Table 7. Here, the charges are the highest for route 1 since it costs the network $6\,\text{$/min}$.

**Table 7** Breakeven charges of Table 6

<table>
<thead>
<tr>
<th>Route</th>
<th>Breakeven charges, $h_{\text{breakeven}}$ ($$/veh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2827</td>
</tr>
<tr>
<td>2</td>
<td>0.0649</td>
</tr>
<tr>
<td>3</td>
<td>0.0695</td>
</tr>
</tbody>
</table>

Next, we consider the example presented in Figure 6. Here, the road capacities are: $L_1 = 100\,\text{veh/min}$, $L_2 = 60\,\text{veh/min}$, $L_3 = 40\,\text{veh/min}$, $L_4 = 50\,\text{veh/min}$ where each minute of delay costs the network the following charges: $c_1 = 1.2\,\text{$/veh/min}$, $c_2 = 0.3\,\text{$/veh/min}$, $c_3 = 0.32\,\text{$/veh/min}$, $c_4 = 0.15\,\text{$/veh/min}$. The tolerable bounds on the private cost of the routes are listed in Table 8. The simulation results appear in the table show one bottleneck since all the queues are empty except that of arc 4. In this example, the simulated optimal rates are the same as the analytical rates, where we have six variables: $x_1, x_2, x_3, q_4^1, q_4^2, q_4^3$ and six corresponding equations. Note that such scenario will cost the network $47\,\text{$/min}$, however, the choice of the
tolerable upper bounds will determine the corresponding optimal rates, queues and charges.

**Figure 6** A transportation network with four arcs and four routes

![Transportation Network Diagram](image)

**Table 8** The results of the configuration in Figure 6, i.e., the steady state rates and queues

<table>
<thead>
<tr>
<th>Route (i)</th>
<th>Rate (veh/min)</th>
<th>Cost components</th>
<th>Vehicles in queues</th>
<th>Cost, $/min</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>C_{\text{max}}</td>
<td>c_i</td>
<td>C_{\text{max}} / c_i</td>
<td>q_i</td>
</tr>
<tr>
<td>1</td>
<td>75/13</td>
<td>15</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>250/13</td>
<td>50</td>
<td>0.6</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>125/13</td>
<td>25</td>
<td>0.3</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>200/13</td>
<td>40</td>
<td>0.15</td>
<td>-</td>
</tr>
<tr>
<td>Network totals</td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

It was found that the upper bound $C_{\text{max}}$ as well as the delay cost $c_i$ will significantly affect the optimal rates, where they can be seen as a measure for the route need to a source. As a result, the higher the value of $C_{\text{max}} / c_i$ the higher the rate of route $i$. The breakeven points for the above scenario are shown in Table 9. The results show clearly that route 1 has to be charged higher than the others since it passes through the four arcs.

**Table 9** Breakeven points of Table 8

<table>
<thead>
<tr>
<th>Route</th>
<th>Breakeven charges, h_{\text{breakeven}} ($/veh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.120</td>
</tr>
<tr>
<td>2</td>
<td>0.780</td>
</tr>
<tr>
<td>3</td>
<td>0.832</td>
</tr>
<tr>
<td>4</td>
<td>0.390</td>
</tr>
</tbody>
</table>

In our last experiment, we consider the previous example with the same mentioned parameters except for the network private cost, where $c_i = 0.3$. After conducting a simulation experiment on such scenario, it was noticed that there are two bottlenecks, namely, arc 3 and arc 4, this because of the change of $C_{\text{max}} / c_i$ level. The simulation showed that the optimal rates converge to the given values in Table 10. For instance, the sum of the rates at arc 3 equals the capacity of that arc, i.e., $x_1 + x_2 + x_3 = 40$ veh while at arc 4 we have $\sum_{i=1}^{4} x_i = 50$. For the queues, we have $\sum_{j=4}^{i} q'_j = C_{\text{max}}$, $\forall i = 1, \ldots, 4$ while the proportionality presents five equalities, specifically, $x_i / x_{i+1} = q'_i / q'_j$, $\forall i = 1, 2$ and $x_i / x_{i+1} = q'_i / q'_k$, $\forall i = 1, 2, 3$. 

Table 10  Steady state rates and queues

<table>
<thead>
<tr>
<th>Route (i)</th>
<th>Rate (veh/min)</th>
<th>$C_i^{\text{max}}$</th>
<th>$c_i$</th>
<th>$C_i^{\text{max}} / c_i$</th>
<th>$q'_1$</th>
<th>$q'_2$</th>
<th>$q'_3$</th>
<th>$q'_4$</th>
<th>Total $\rho'_i$</th>
<th>Cost, $\rho'_i$ ($/\text{min}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>60/9</td>
<td>18</td>
<td>1.2</td>
<td>15</td>
<td>0</td>
<td>0</td>
<td>15/9</td>
<td>120/9</td>
<td>15</td>
<td>18</td>
</tr>
<tr>
<td>2</td>
<td>200/9</td>
<td></td>
<td></td>
<td></td>
<td>-</td>
<td>0</td>
<td>50/9</td>
<td>400/9</td>
<td>50</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>100/9</td>
<td>8</td>
<td>0.32</td>
<td>25</td>
<td>-</td>
<td>-</td>
<td>25/9</td>
<td>200/9</td>
<td>25</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>6</td>
<td>0.3</td>
<td>20</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>20</td>
<td>20</td>
<td>6</td>
</tr>
<tr>
<td>Network totals</td>
<td></td>
<td>10</td>
<td>100</td>
<td>110</td>
<td>100</td>
<td>110</td>
<td>47</td>
<td></td>
<td>$47$</td>
<td></td>
</tr>
</tbody>
</table>

The previous example shows that although the same total network private cost is the same (=$47$), however, the network had to change the rates according to the changes in $C_i^{\text{max}} / c_i$. In addition, the network can control its levels of service for each route by controlling the upper bounds on the private costs. As illustrated in previous sections, the cost upper bounds represent the tolerable limits paid by the network to provide transportation service while maximising the global benefits of the network routes (i.e., users).

Table 11 presents the breakeven points. It is worth notifying that although the network has to spend $47/min for the operation of the routes in both scenarios for the configuration in Figure 6, however, two different flow patterns resulted to breakeven this quantity.

Table 11  Breakeven points of Table 10

<table>
<thead>
<tr>
<th>Route (i)</th>
<th>Breakeven charges, $h_{\text{breakeven}}$ ($/\text{veh}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.700</td>
</tr>
<tr>
<td>2</td>
<td>0.675</td>
</tr>
<tr>
<td>3</td>
<td>0.720</td>
</tr>
<tr>
<td>4</td>
<td>0.600</td>
</tr>
</tbody>
</table>

7 Conclusions

A cost driven traffic assignment model was proposed to maximise the global benefit of transportation network users. The model was extended to its primal-dual problem where the projection algorithm was used to solve for the optimal traffic rates while subjecting the objective function to the bounds of the network tolerable costs. A timely synchronised iterative solution was used to solve the model in a real time simulated environment. The private cost upper bounds were found to be crucial for determining the traffic flow rates, where higher per-route bounds resulted in the acquisition of higher traffic rates. Further, the value of the private costs can significantly affect the level of vehicles in transit. Smaller cost values result in higher expected traffic rates and hence more queues and traffic jams. For a successful operation of the transportation network, the affordable upper bounds on the costs has to be set carefully in order to maximise the users benefits and to avoid extreme queues and delays. At the bottlenecks, the optimal rate ratios were
shown to be proportional to the ratios of queues keeping acceptable allocation among the routes. Additionally, the optimal rates could bring the bottleneck roads to full utilisation while keeping steady intensity of traffic in transit. A breakeven analysis was also presented to estimate the network charges. The charges were estimated according to the network optimal flows and bounds. It was found that the longer the route, the higher its charge. Breakeven surcharges represent the minimum quantities a network charges from its commuters for a profitable operation.

The presented method has inspiring resource allocation characteristics and reveals a greater control on the queues in a heterogenous transportation network. Moreover, the exact solution of any network can be obtained analytically after identifying the bottlenecks, however, the number of variable can increase significantly for larger networks, hence, real-time simulation remains an effective tool to identify the optimal rates. More interestingly, the algorithm does not necessitate any correspondence between the different routes or the admission to any global network parameters; rather, the solution can be reached in parallel through the network routes. The model was verified by conducting extensive simulation experiments. The results could demonstrate an attractive convergence, execution and sound resource allocation between the shared resources while accomplishing global optimality in a parallel formation.

References
Cost driven traffic assignment in transportation networks


