UNIFORM CONVEXITY OF KÖTHE–BOCHNER FUNCTION SPACES

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Abstract. Of concern are the Köthe–Bochner function spaces $E(X)$, where $X$ is a real Banach space. Thus, of concern is the uniform convexity on the Köthe–Bochner function space $E(X)$. We show that $E(X)$ is uniformly convex if and only if both spaces $E$ and $X$ are uniformly convex. It is the uniform convexity that is the focal point.

1. Basic Concepts

Throughout this paper $(T, \Sigma, \mu)$ denote a δ-finite complete measure space and $L^\circ = L^\circ(T)$ denotes the space of all (equivalence classes) of $\Sigma$-measurable real valued functions. For $f, g \in L^\circ, f \leq g$ means that $f(t) \leq g(t)$ $\mu$-almost everywhere $t \in T$.

A Banach space is said to be a Köthe space if:

1. For any $f, g \in L^\circ, |f| \leq |g|, g \in E$ imply $f \in E$ and $\|f\|_E \leq \|g\|_E$.
2. For each $A \in \Sigma$, if $\mu(A)$ is finite then $\chi_A \in E$. See [12, p. 28].

Let $E$ be a Köthe space on the measure space $(T, \Sigma, \mu)$ and $(X, \|\cdot\|_X)$ be a real Banach space. Then $E(X)$ is the space (of all equivalence classes of) strongly measurable functions $f : T \to X$ such that $\|f(\cdot)\|_X \in E$ equipped with the norm

$$\|\|f\|\| = \|\|f(\cdot)\|_X\|_E.$$

The space $(E(X), \|\|\|_E)$ is a Banach space called the Köthe–Bochner function space [11, p. 147]. The most important class of Köthe–Bochner function spaces $E(X)$ are the Lebesgue–Bochner spaces $L^p(X), (1 \leq p < \infty)$ and their generalization the Orlicz–Bochner spaces $L^\phi(X)$. They have been studied by many authors [1], [2], [6], [7]. The geometric properties of the Köthe–Bochner function spaces have been studied by many authors, (e.g. [1], [5], [10]).
A Banach space is uniformly convex if and only if for every \( \epsilon > 0 \), there is a unique \( \delta > 0 \) such that for all \( x, y \) in \( X \), the conditions \( \|x\| = \|y\| = 1 \) and \( \|x - y\| > \epsilon \) imply \( \|\frac{x+y}{2}\| < 1 - \delta \). Moreover this definition is equivalent to the following, [3, p. 127], for every pair of sequences \((x_n)\) and \((y_n)\) in \( X \) with \( \|x_n\| \leq 1, \|y_n\| \leq 1 \) and \( \|x_n + y_n\| \leq 2 \), it follows \( \|x_n - y_n\| \to 0 \).

For a Banach space \( X \), we denote by \( \delta_X(\epsilon) \) the modulus of convexity
\[
\delta_X(\epsilon) = \inf \left\{ 1 - \frac{1}{2} \|x + y\| : x, y \in X, \|x\| = \|y\| = 1, \|x - y\| > \epsilon \right\}
\]
for any \( \epsilon \in [0, 2] \). Note that \( X \) is uniformly convex if and only if \( \delta_X(\epsilon) > 0 \) whenever \( \epsilon > 0 \). If \( X \) is uniformly convex, we define the characteristic of convexity of by
\[
\epsilon_r(X) = \sup \{ \epsilon \in [0, 2] : \delta_X(\epsilon) \leq r \}.
\]

The uniform convexity of both the Lebesgue–Bochner spaces \( L^p(X), (1 \leq p < \infty) \) and the Orlicz–Bochner spaces \( L^\Phi(X) \) have been studied by many authors (e.g. see [4], [5], [8], [9], [14]).

In [14], M. Smith and B. Truett showed that many properties akin to uniform convexity lift from \( X \) the Lebesgue–Bochner spaces \( L^p(X), (1 \leq p < \infty) \). A survey of rotundity notions in the Lebesgue–Bochner spaces \( L^p(X), (1 \leq p < \infty) \) and sequences spaces can be found in [13].

In [10], A. Kaminska and B. Truett showed that many properties akin to uniform convexity lift from \( X \) to \( E(X) \). The approach used in their paper is different than that we used.

This paper is devoted to the study of uniform convexity of the Köthe–Bochner function space \( E(X) \), where \( X \) is a real Banach space.

## 2. Main Results

Let us prove some preliminary results which will allow us in Theorem 4 to obtain a characterization of the uniform convexity of the Köthe–Bochner function space \( E(X) \).

**Lemma 1.** If \((f_n)\) and \((g_n)\) are sequences in the Köthe–Bochner function space \( E(X) \) with \( \|f_n\| = \|g_n\| = 1 \) and \( \|f_n + g_n\| \to 2 \) then
\[
\|f_n(\cdot)\|_X + \|g_n(\cdot)\|_X \to 2.
\]

**Proof.** Using the following inequalities we get the required result
\[
\|f_n + g_n\| = \|f_n(\cdot) + g_n(\cdot)\|_E \leq \|f_n(\cdot)\|_X + \|g_n(\cdot)\|_X \leq \|f_n\| + \|g_n\| = 2.
\]
Lemma 2. Let $X$ be a real Banach space and $E$ be a Köthe space. If $(E, \| \cdot \|_E)$ is uniformly convex Köthe space and $(f_n), (g_n)$ are sequences in the Köthe–Bochner function space $E(X)$ with $\| f_n \| = \| g_n \| = 1$ and $\| f_n + g_n \| \to 2$ then,

(i) $\| f_n (\cdot) \|_X - \| g_n (\cdot) \|_X \|_E \to 0.$

(ii) $\| f_n (\cdot) \|_X + \| g_n (\cdot) \|_X - \| f_n (\cdot) + g_n (\cdot) \|_X \|_E \to 0.$

Proof. Note that $(\| f_n (\cdot) \|_X)$ and $(\| g_n (\cdot) \|_X)$ are sequences in $E$ with

$\| f_n (\cdot) \|_X \|_E = \| g_n (\cdot) \|_X \|_E = 1.$

Using Lemma 1 and the fact that $E$ is uniformly convex Köthe space, it is straightforward to get (i). To prove (ii) we note first the inequalities

$$2 \| f_n + g_n \| = 2 \| f_n (\cdot) + g_n (\cdot) \|_E$$

$$\leq \| f_n (\cdot) \|_X + \| g_n (\cdot) \|_X + \| f_n (\cdot) + g_n (\cdot) \|_E$$

$$\leq 2 \| f_n (\cdot) \|_X + \| g_n (\cdot) \|_X$$

$$\leq 2 (\| f_n \| + \| g_n \|) = 4.$$

In this line, we get

$$\left\| \frac{f_n (\cdot) + g_n (\cdot)}{2} \right\|_X + \frac{\| f_n (\cdot) \|_X}{2} + \frac{\| g_n (\cdot) \|_X}{2} \to 2.$$

Due to the uniform convexity of $E$ and the facts that

$\left\| \frac{f_n (\cdot) + g_n (\cdot)}{2} \right\|_X \leq 1$ and $\left\| \frac{\| f_n (\cdot) \|_X}{2} \right\| + \left\| \frac{\| g_n (\cdot) \|_X}{2} \right\|_E \leq 1,$

result (ii) is completely proved.

Lemma 3. Let $X$ be a uniformly convex Banach space and $E(X)$ be a Köthe–Bochner function space. If $f, g \in E(X)$ then

$$\| f - g \| = \epsilon_r (X)(\| f \| + \| g \|) + \frac{2}{\epsilon} (\| f (\cdot) \|_X - \| g (\cdot) \|_X \|_E +$$

$$\| f (\cdot) \|_X + \| g (\cdot) \|_X - \| f (\cdot) + g (\cdot) \|_X \|_E.$$
\[ \geq 1 + \frac{\|f(t)\|_X - \epsilon \|f(t)\|_X}{\|f(t)\|_X} - \epsilon \]
\[ = 2 - 2\epsilon \]
and, since \( X \) is uniformly convex and \( \|f(t)\|_X, \|g(t)\|_X \) are elements of \( X \) for a fixed \( t \in T \), find that
\[ \|\tilde{f}(t) - \tilde{g}(t)\|_X \leq \epsilon r(X). \]

Therefore we deduce that
\[ (1) \quad \|f(t) - g(t)\|_X \leq \epsilon r(X) \max \{\|f(t)\|_X, \|g(t)\|_X\}. \]

Case 2. (i) is not true. We imply that if
\[ \|f(t)\|_X - \|g(t)\|_X \geq \max \{\|f(t)\|_X, \|g(t)\|_X\}. \]
And so
\[ (2) \quad \|f(t) - g(t)\|_X \leq 2 \max \{\|f(t)\|_X, \|g(t)\|_X\} \]
\[ < \frac{2}{\epsilon} \|f(t)\|_X - \|g(t)\|_X \]. \]

Case 3. (ii) is not true. We can get that
\[ (3) \quad \|f(t) - g(t)\|_X \leq 2 \max \{\|f(t)\|_X, \|g(t)\|_X\} \]
\[ < \frac{2}{\epsilon} (\|f(t)\|_X + \|g(t)\|_X - \|f(t) + g(t)\|_X). \]

Inequalities (1), (2), and (3) give the inequality
\[ \|f(t) - g(t)\|_X \leq \epsilon r(X)(\|f(t)\|_X + \|g(t)\|_X) + \frac{2}{\epsilon} (\|f(t)\|_X + \|g(t)\|_X - \|f(t) + g(t)\|_X) \]
and (ii) is completely proved. \( \square \)

In this line, we are able to introduce the following main theorem about the uniform convexity of the Köthe–Bochner function space \( E(X) \).

**Theorem 4.** Let \( X \) be a real Banach space and \( E \) be a Köthe space. Then \( E(X) \) is a uniformly convex Köthe–Bochner space if and only if both \( X \) and \( E \) are uniformly convex.

**Proof.** Suppose \( E(X) \) is uniformly convex Köthe–Bochner function space. Since both spaces \( X \) and \( E \) are embedded isometrically into \( E(X) \), and due to the fact that uniform convexity inherited by subspaces, we deduce that both spaces \( X \) and \( E \) are uniformly convex.

Conversely, suppose \( X \) and \( E \) are uniformly convex spaces. Let \( (f_n) \) and \( (g_n) \) be sequences in \( E(X) \) with \( \|f_n\| = \|g_n\| = 1 \) and \( \|f_n + g_n\| \to 2 \), then by Lemma 2 it follows that
\[ \|\|f_n(\cdot)\|_X - \|g_n(\cdot)\|_X\|_E \to 0 \]
and 

$$\| f_n(\cdot) \|_X + \| g_n(\cdot) \|_X - \| f_n(\cdot) + g_n(\cdot) \|_X \|_E \to 0.$$ 

Take $\epsilon^* > 0$ and choose $\delta > 0$ such that $\epsilon_r(X) \leq \frac{\epsilon^*}{4}$. Choose $K$ sufficiently large so that for all $n > K$, 

$$\| f_n(\cdot) \|_X + \| g_n(\cdot) \|_X \|_E + \| f_n(\cdot) \|_X + \| g_n(\cdot) \|_X \|_E < \frac{1}{4} \epsilon \epsilon^*.$$ 

From Lemma 3 it follows that 

$$\| f_n - g_n \| = \epsilon_r(X)(\| f_n \| + \| g_n \|) + \frac{1}{\epsilon}(\| f_n(\cdot) \|_X + \| g_n(\cdot) \|_X \|_E \|_E)$$ 

$$+ \frac{1}{\epsilon}(\| f_n(\cdot) \|_X + \| g_n(\cdot) \|_X \|_E \|_E)$$ 

$$\leq 2 \epsilon_r(X) + \frac{1}{\epsilon}(\epsilon^* \epsilon^*)$$ 

$$\leq \frac{1}{2} \epsilon^* + \frac{1}{2} \epsilon^* = \epsilon^*,$$ 

for all $n > K$.

Consequently $\| f_n - g_n \| \to 0$, which means that $E(X)$ is a uniformly convex Köthe–Bochner space, and thereby the theorem is completely proved. □

References


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