SOME RESULTS ON CONNECTED AND MONOTONE FUNCTIONS

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Abstract. A function \( f : X \to Y \) is called connected if \( f(C) \) is connected for each connected subset \( C \) of \( X \); and \( f \) is called monotone if \( f^{-1}(C) \) is connected for each connected subset \( C \) of \( Y \). In this paper, we prove some results of connected and monotone functions. Conditions on domain and/or range implying continuity of connected and monotone functions are obtained.

1. Introduction

The class of connected and monotone functions was introduced by Whyburn in 1934. Some important results are given on connected and monotone functions ([2], [7]). In ([3], [4], [5]) M. R. Hagan gave some results on which monotone and/or connected functions are continuous by assuming for the domain and/or range various combinations of properties. The concepts of connected functions and monotone functions are independent of each other. Moreover, connected functions and monotone functions are weak types of continuous functions and open functions respectively. However, conditions on the domain and/or range of functions can imply that connected or monotone functions are continuous. In this paper, we prove some results of connected and monotone functions.

2. Preliminary definitions and results

In this section we introduce some definitions and results that we will use.

Definition 1 ([7]). Let \( X = (X, \tau_1) \) and \( Y = (Y, \tau_2) \) be two topological spaces let \( f : X \to Y \) be a function Then the function \( g : X \to X \times Y \) defined by \( g(x) = (x, f(x)) \) is called the graph function.

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Definition 2 ([7]). Let $X = (X, \tau)$ be a topological space. Then

(i) $X$ is said to be biconnected space if given two nonempty disjoint connected sets whose union is $X$, one of them must be a singleton.

(ii) $X$ is said to be widely connected if every nondegenerate connected subset is dense.

Theorem 3 ([7]). Let $X = (X, \tau_1)$ be any topological space, $Y = (Y, \tau_2)$ be a $T_1$-space and $f : X \rightarrow Y$ be a connected function. If $K \subseteq Y$ has closed components in $Y$, then $f^{-1}(K)$ has closed components in $X$.

Theorem 4 ([7]). Let $f : X \rightarrow Y$ be a connected function from a locally connected space $X$ into a $T_1$-space $Y$. If

(i) $Y \setminus K = A \cup B$ where $A$ and $B$ are two nonempty disjoint open sets, and

(ii) $f^{-1}(K)$ is a connected subset of $X$,

then the sets $f^{-1}(A \cup K)$ and $f^{-1}(B \cup K)$ are connected.

Theorem 5 ([7]). Let $f : X \rightarrow Y$ be a connected function from a locally connected space $X$ into a $T_1$-space $Y$. Moreover, let $Y \setminus K = A \cup B$ where $A$ and $B$ are two nonempty disjoint open sets. If $\{K_n\}$ is a sequence of sets contained in $Y$ satisfying the conditions

(i) $Y \setminus K_n = A_n \cup B_n$ where $A_n$ and $B_n$ are two nonempty disjoint open sets,

(ii) $f^{-1}(K_n)$ is a connected subset of $X$ and

(iii) $A_{n-1} \cup B_{n-1} \subseteq A_n \cup B_n$,

then $CL( f^{-1}( \bigcup_{n=1}^{\infty} (A_n \cup K_n)) ) \subseteq f^{-1}(A \cup K)$.

Theorem 6 ([7]). Let $X = (X, \tau_1)$ be any topological space, $Y = (Y, \tau_2)$ be a $T_1$-space and let $f : X \rightarrow Y$ be a connected function. If $K \subseteq Y$ has closed components in $Y$, then $f^{-1}(K)$ has closed components in $X$.

3. Some results on Connected Monotone Functions

Theorem 7. If $X$ is widely connected, $f : X \rightarrow Y$ is a monotone connected function, and $Y$ is a $T_1$-space; then either $f$ is injective or $f$ is constant.

Proof. Suppose $X$ is widely connected and $f : X \rightarrow Y$ is monotone connected function. Then if $f$ is not injective, there exists $\alpha \in Y$ with $|f^{-1}(\{\alpha\})| > 1$. But $X$ is widely connected, then $f^{-1}(\{\alpha\}) = X$. Now since $\{\alpha\}$ is closed in $Y$, then by theorem 6 $f^{-1}(\{\alpha\})$ has closed components in $X$ which implies $f^{-1}(\{\alpha\})$ is closed since $f^{-1}(\{\alpha\})$ is connected. Hence, $f^{-1}(\{\alpha\}) = f^{-1}(\{\alpha\}) = X$ which implies that $f$ is constant.

Corollary 8. If $X$ is widely connected and $f : X \rightarrow \mathbb{R}$ is a monotone connected function, then either $f$ is injective or $f$ is constant.
**Theorem 9.** A function \( f : X \to Y \) is monotone if and only if the graph function \( g \) of \( f \) is monotone.

**Proof.** Suppose \( f : X \to Y \) is monotone and \( A \) is a connected subset of \( X \times Y \); then \( p_Y(A) \) is a connected subset of \( Y \) since \( p_Y \) is continuous. This implies that \( f^{-1}(p_Y(A)) \) is connected subset of \( X \) since \( f \) is monotone, but \( g^{-1}(A) = f^{-1}(p_Y(A)) \). Hence, \( g^{-1}(A) \) is connected and therefore \( g \) is monotone. Conversely, suppose \( g \) is monotone and \( C \) is a connected subset of \( Y \). Now \( p_Y^{-1}(C) \) is a connected subset of \( Y \) since \( p_Y \) is open. This implies that \( g^{-1}(p_Y^{-1}(C)) \) is a connected subset of \( X \) but \( f^{-1}(C) = g^{-1}(p_Y^{-1}(C)) \). Hence, \( f^{-1}(C) \) is connected and therefore \( f \) is monotone. \( \Box \)

**Theorem 10.** A function \( f : X \to Y \) is connected if and only if the graph function \( g \) of \( f \) is connected.

**Theorem 11.** Let \( f : X \to Y \) be a bijective monotone connected function. Then \( X \) is biconnected if and only if \( Y \) is biconnected.

**Proof.** Let \( f : X \to Y \) be a bijective monotone connected function. Suppose \( X \) is biconnected and \( A, B \) be two disjoint connected subsets of \( Y \), with \( Y = A \cup B \). Then \( f^{-1}(A) \) and \( f^{-1}(B) \) are two disjoint connected subsets of \( Y \), with \( Y = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \) because \( f \) is a bijective monotone function. Then either \( f^{-1}(A) \) or \( f^{-1}(B) \) is a singleton because \( X \) is biconnected. Hence, either \( A \) or \( B \) is a singleton and therefore the result follows. Conversely, suppose \( Y \) is biconnected and let \( A, B \) be two disjoint connected subsets of \( X \), with \( X = A \cup B \). Then \( f(A) \) and \( f(B) \) are two disjoint connected subsets of \( Y \), with \( Y = f(A \cup B) = f(A) \cup f(B) \) because \( f \) is a bijective connected function. Then either \( f(A) \) or \( f(B) \) is a singleton because \( Y \) is biconnected. Hence, either \( A \) or \( B \) is a singleton. This completes the proof of this theorem. \( \Box \)

**Theorem 12.** Let \( f : X \to \mathbb{R} \) be a monotone connected function. If \( \{\alpha_n\} \) is an increasing sequence and \( \{\beta_n\} \) is a decreasing sequence in \( \mathbb{R} \) with \( \alpha_n < \beta_n \) for all \( n \) such that \( \alpha_n \to \alpha \) and \( \beta_n \to \beta \), then \( f^{-1}([\alpha, \beta]) \subset f^{-1}((\alpha, \beta]) \).

**Proof.** Suppose \( K = [\alpha, \beta] \), \( K_n = [\alpha_n, \beta_n] \), \( A_n = (-\infty, \alpha_n) \), and \( B_n = (\beta_n, \infty) \). Then it is clear that the following conditions are satisfied

(i) \( \mathbb{R} \setminus K_n = A_n \cup B_n \) where \( A_n \) and \( B_n \) are two nonempty disjoint open sets for \( n = 1, 2, \ldots \)

(ii) \( A_{n-1} \cup B_{n-1} \subset A_n \cup B_n \) for \( n = 1, 2, \ldots \)

(iii) \( f^{-1}(K_n) \) is connected for for \( n = 1, 2, \ldots \) since \( K_n \) is connected and \( f \) is monotone.

Using theorem 5, we have \( f^{-1}([\alpha, \beta]) \subset f^{-1}((\alpha, \beta]) \). \( \Box \)

**Theorem 13.** Let \( f : X \to Y \) be a monotone connected function from a space \( X \) to a \( T_1 \)-space \( Y \). If \( V \subset Y \) is open and \( Y - V \) is connected, then \( f^{-1}(V) \) is open.
Proof. Suppose \( V \subset Y \) is open and \( Y - V \) is connected. Then \( Y - V \) is a closed connected subset of \( Y \), and therefore \( f^{-1}(Y - V) = X - f^{-1}(V) \) is connected and has closed components by Theorem 6. Thus the connectedness of the set \( f^{-1}(Y - V) = X - f^{-1}(V) \) implies that \( X - f^{-1}(V) \) is closed. Hence, \( f^{-1}(V) \) is open. \( \Box \)

**Corollary 14.** If \( f : X \rightarrow \mathbb{R} \) is a monotone connected function, then \( f^{-1}(-\infty, \alpha) \) and \( f^{-1}(\alpha, \infty) \) are open for all \( \alpha \in \mathbb{R} \).

**Corollary 15.** If \( f : X \rightarrow \mathbb{R} \) is a monotone connected function, then \( f \) is continuous.

**Corollary 16.** If \( f : X \rightarrow S^1 \) is a monotone connected function, then \( f^{-1}(V) \) is open for any open connected subset of \( S^1 \).

**Corollary 17.** If \( f : X \rightarrow S^1 \) is a monotone connected function, then \( f \) is continuous.

**References**


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