Stress distribution in granular heaps using multi-slip formulation

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SUMMARY

Many experiments on conical piles of granular materials have indicated, contrary to simple intuition that the maximum vertical stress does not occur directly beneath the sand-pile vertex but rather at some distance from the apex resulting in a ring of maximum vertical stress. Some recent experiments have shown that the observed stress dip is very much dependent on construction history. A multi-slip model has been proposed to investigate the stress dip phenomenon in granular heaps. The double-slip version of the model was implemented into ABAQUS and used to study the vertical stress distribution along the base of a granular pile. The numerical simulations show that plastic deformation is confined within the localized region around the apex while the rest of the pile is in an elastic state of deformation. Within the plastic region the stress distribution differs significantly depending on the initial active slip orientation. The results show that for homogenous state of granular materials such as those produced by a raining procedure the vertical stress profile along the base reached its peak at the apex (i.e. no dip was observed). On the contrary, granular heaps constructed by the use of a localized source such as a funnel resulted in a significant reduction in the stress distribution within the ring with the minimum attained beneath the peak (i.e. a dip). Therefore, we believe that the initial microstructure and thus the initial slip orientation resulting from sand deposition is the source of the stress dip phenomenon. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: granular; stress; dilatancy; friction; plastic; microstructure slip

INTRODUCTION

The description of the mechanical behaviour of granular materials, such as sands, clays, and powders is important to many fields of science and engineering. Granular assemblies are intriguing systems rich with unusual properties such as dilatancy, arching, instability, and

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thixotropy. These properties conspire to create a complex system with numerous instabilities. Examples include liquefaction failures, density waves in hopper flows, and stick-slip motion in shear flows. There is currently a wide range of mathematical models available for describing the mechanical behaviour of granular materials. Most of them are, however, phenomenological and are often based on simple ideas such as Coulomb frictional sliding, with a single friction coefficient and a single slip surface. These models do not provide a satisfactory solution to many of the problems in granular materials.

One of the intriguing problems that have captured the interest of engineers and scientists is the vertical stress distribution at the base of a granular heap. Many experiments on conical piles of granular materials have indicated, contrary to simple intuition that the maximum vertical stress distribution does not occur directly beneath the sand-pile vertex but rather at some distance from the apex location resulting in a ring of maximum vertical stress. The literature is now replete with experiments and simulations examining the profiles under static piles of granular materials (See References [1, 2]). The dramatic dip in the stress reported by Smid and Novosad [3] promoted many physicists to examine this problem in detail. Such studies have brought forth conflicting results and competing constitutive models for explaining the observed behaviour. While some studies have shown that depending on the construction history, little or no stress dip was observed [4], in others the dip was significant [1, 5, 6].

Vanel et al. [4] have studied the effects of construction procedures on the pressure profile using two different methods to construct granular heaps. In the first, a funnel was used whilst in the second a sieve was used to rain the sand and construct the heaps. The first procedure was termed the localized source procedure; the second the ‘raining procedure’. It was found that a pressure dip at the centre of the pile was observed only if the localized source was used. In the case of a more uniformly vertical filling via a raining procedure with a fixed pouring height, the dip in stress was not observed.

A heuristic explanation of the mechanism producing the dip was proposed by Vanel et al. [4]. Based on past observation of force chains on photoelastic disks it was hypothesized that during the localized source procedure, particles formed stress chains oriented preferentially in the direction of the slope (Figure 1). It was found that these chains form arches that shield the centre from some of the weight, thereby forming the dip.

Figure 1. Two-dimensional pile of photoelastic disks (diameters 0.74 and 0.9 cm) created by a localized-source procedure. The centre section of the image, with a height of ~ 30 cm, is viewed between crossed polarizers, allowing one to see the underlying stress structure (after Reference [4]). (Reprinted Figure 1 with permission from Memories in sand: Experimental tests of construction history on stress distributions under sandpiles, L. Vanel, D. Howell, D. Clark, R. P. Behringer and E. Clément, Phys. Rev. E, 60:R5040–R5043, 1999. DOI:10.1103/PhysRevE.60.R5040 Copyright © 1999 American Physical Society.)

It is evident from the experiments of Reference [4] that localization is the main source of the dip in stress. Therefore, efforts using classical Sokolovski type rigid plastic continuum formulations to account for this effect would entail the use of unrealistic values of $90^\circ$ for granular material friction angle [7]. It is noted that some elastic–plastic models have predicted the existence of the local stress minimum at the centre [8]. However, such studies cannot predict the effect of the preparation method on the observed dip without accounting for localization.

Anand and Gu [9] have considered the stress dip as problem of strain localization and predicted the dip in stress observed by Brockbank et al. [6] using a fixed double slip formulation. Al Hattamleh et al. [10] have shown that the orientation of the initial slip system is dependent on the microstructure and test arrangement and conditions of the specimen. They have proposed a multi-slip formulation that accounts for initial microstructure of granular materials. This model is used here to examine the conditions that lead to the vertical stress dip phenomenon in granular heaps.

**MATHEMATICAL PRELIMINARIES**

The velocity gradient is split into two parts; symmetric and skew-symmetric. The symmetric part represents the pure stretching tensor, $D_{ij}$, and the skew symmetric part represents the spin tensor, $W_{ij}$:

$$ D_{ij} = \frac{1}{2} (L_{ij} + L_{ji}^T) $$

$$ W_{ij} = \frac{1}{2} (L_{ij} - L_{ji}^T) $$

The stretching rate $D_{ij}$ can be decomposed as

$$ D_{ij} = D_{ij}^e + D_{ij}^p $$

where $D_{ij}^e$ and $D_{ij}^p$ are the elastic and plastic parts, respectively. Likewise, the spin tensor is written as

$$ W_{ij} = \omega_{ij} + W_{ij}^p $$

where $\omega_{ij}$ is the spin of microstructure and $W_{ij}^p$ is the plastic spin.

$D_{ij}$ can be split into a volumetric strain rate ($D$), and a deviatoric strain rate $d_{ij}$ as

$$ D = D_{mm} \quad d_{ij} = D_{ij} - \frac{1}{3} D \delta_{ij} $$

The equation of equilibrium for quasi-static loading conditions is given by

$$ \sigma_{ij,j} + b_i = 0 $$

where $\sigma_{ij}$ is the Cauchy stress tensor and $b_i$ is the body force per unit volume.

The elastic rate of stretching, $D_{ij}^e$ is assumed to follow Hooke’s law:

$$ \delta_{ij}^e = C_{ijkl}^e D_{kl}^e $$

$$ C_{ijkl}^e = G \left( \delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl} + \frac{2v}{1 - 2v} \delta_{ij} \delta_{kl} \right) $$

where $C_{ijkl}^e$ is the elasticity tensor, $G$ is the shear modulus, $v$ is the Poisson’s ratio. The corotational rate of Cauchy stress tensor $\tilde{\sigma}_{ij}$ is defined with respect to frame rotating with the material:

$$ \tilde{\sigma}_{ij} = \sigma_{ij} - \omega_{ik} \sigma_{kj} + \sigma_{ik} \omega_{kj} $$

MULTI-SLIP MODEL FOR PLASTIC FLOW

A typical microscopic shear plane in a granular material is shown in Figure 2. In the multi-slip model for soils, plastic deformation is viewed in terms of slips on two planes-defined by their normal unit vector \( n_i \) and slip direction \( m_i \). It is noted that while in polycrystalline metals slip systems are uniquely defined, the definition of slip planes in soils is not clearly defined. Therefore, it is necessary to choose a set of active planes. In most cases, they can be defined either with respect to the maximum shear stress plane or in terms of the pre-existing weak planes of shear which would depend upon the microstructure as will be explained later.

The approach adopted here is a generalization of the ‘double-shearing’ plane strain constitutive model [9,11–16]. The plane strain model is generalized to three dimensions including the effects of elastic deformation, the typical pressure sensitive and dilatant hardening/softening response observed in granular materials, as well as the effect of heterogeneous microstructure. The dilation rate with respect to the shear plane is given as a combined function of the effective plastic strain on the slip systems and of a dilatancy coefficient. The dilatancy coefficient is also dependent on the effective plastic strain. These ideas form the basis of the constitutive model developed here for granular materials.

An idealized double slip system is shown in Figure 3. The components of one of the slip systems \( s^{(1)} \) is given by

\[
\begin{align*}
    m_i^{(1)} &= -\cos \zeta_1 \cdot e_1 - \sin \zeta_1 \cdot e_2 \\
    n_i^{(1)} &= \sin \zeta_1 \cdot e_1 - \cos \zeta_1 \cdot e_2
\end{align*}
\]  

Figure 2. Shearing plane in a collection of granular particles.
where $\zeta$ is the angle measured with respect to the minor principal stress axis and $e_1, e_2$ are the unit vectors in the Cartesian coordinate system.

The plastic strain rate tensor is made up of simple shearing strain rates $\dot{\gamma}_p^{(s)}$ on each of the slip system $s$, and this shearing is accompanied by shear-induced dilatancy rate $\dot{\nu}^{(s)}$ in the directions normal to the shear directions:

$$
D_{ij}^p = \sum_{s=1}^{S} \dot{\gamma}_p^{(s)} M_{ij}^{(s)} + \sum_{s=1}^{S} \dot{\nu}^{(s)} N_{ij}^{(s)}
$$

$$
M_{ij}^{(s)} = (m_{i}^{(s)} n_{j}^{(s)} + n_{i}^{(s)} m_{j}^{(s)})/2
$$

$$
N_{ij}^{(s)} = n_{i}^{(s)} n_{j}^{(s)}
$$

$$
W_{ij}^p = \sum_{s=1}^{S} \dot{\gamma}_p^{(s)} V_{ij}^{(s)}
$$

$$
V_{ij}^{(s)} = (m_{i}^{(s)} n_{j}^{(s)} - n_{i}^{(s)} m_{j}^{(s)})/2
$$

where $D_{ij}^p$ is the plastic strain rate tensor, $W_{ij}^p$ is the plastic spin, $\dot{\gamma}_p^{(s)} = \sqrt{2\dot{\gamma}_p^{(s)} d_{ij}^p d_{ji}^p}$ is the effective plastic strain rate on the $s$th slip plane system, and $\dot{\nu}^{(s)} = \beta_j \dot{\gamma}_p^{(s)}$ is the dilatation rate, with $\beta_j \dot{\gamma}_p^{(s)}$ being the mobilized dilatancy coefficient and $\dot{\gamma}_p^p = \int \dot{\gamma}_p \, dt$. Moreover, it is assumed that the slip planes are oriented at a given angle relative to a fixed plane in space. This can often be the plane of maximum obliquity stress ratio [9,11,12]. However, we believe that this angle is very much dependent on the granular microstructure relating to factors such as
depositional history and angularity. This angle is then treated as a material parameter and the rotation of the slip plane will be evaluated as (assuming that $m_i^{(s)}$ remain normal to $n_i^{(s)}$)

$$\dot{m}_i = \omega_{ij} m_j$$ (12a)

$$\dot{n}_i = \omega_{ij} n_j$$ (12b)

where $\omega_{ij}$ is the spin of microstructure and given by Equation (4).

**Yield functions**

The set of yield functions on the slip systems of the granular material is assumed to follow the gradient criterion, $f^{(s)}$:

$$f^{(s)} = q^{(s)} - \mu p^{(s)} - c_1 \nabla^2 p^{(s)}$$ (13)

where $q^{(s)}$ and $p^{(s)}$ are the resolved shear and normal stresses given as a function of Cauchy stress tensor as

$$q^{(s)} = \sigma_{ij} : M^{(s)}_{ij}$$ (14)

$$p^{(s)} = \sigma_{ij} : N^{(s)}_{ij}$$ (15)

The hardening/softening behaviour due to redistribution of contact is modelled through an evolution law for the mobilized friction coefficient, (e.g. References [9, 17, 18]). $c_1$ represents the first gradient coefficient and $\nabla^2 p^{(s)}$ represents the Laplacian of effective plastic strain.

It is worth noting that the number of independent slip planes is dependent on the problem under investigation. For general three-dimensional deformation of compressible materials the minimum number of conjugate slip systems required is six whilst for incompressible materials it is five. For two-dimensional problems, however, two slip planes are sufficient.

**Effective plastic shear strain rate and stiffness tensors**

Utilizing the yield and consistency conditions $f = 0; \dot{f} = 0$, along with Equations (6), (13), (14), and (15), the plastic strain rate could be evaluated as [19]

$$\dot{\sigma}_{ij} = \frac{(M^{(s)}_{ij} + z \mu \delta_{ij}) : D_{ij} - c_1 \nabla^2 p^{(s)} / G}{1 + |p^{(s)}| h_i / G + z \beta \mu + c_1' \nabla^2 p^{(s)} / G}$$ (16)

where $h_i = \partial \mu / \partial \gamma^p$, is the strain hardening/softening modulus and $c_1' = dc_1 / d\gamma^p$, substituting Equation (16) into Equation (7) and combining it with Equations (10) and letting $c_1' = 0$ leads to

$$\dot{\sigma}_{ij} = C_{ijkl} D_{kl} - \delta_{ij}$$ (17)

$$C_{ijkl} = C_{ijkl}^p - C_{ijkl}^s$$ (17a)

$$\delta_{ij} = \sum_{x=1}^{S} c_1 \nabla^2 p^{(s)} (M^{(s)}_{ij} + z \beta \delta_{ij}) / H^{(s)}$$ (17b)

where $H^{(s)} = 1.0 + |p^{(s)}| h_i / G + z \beta \mu$, and $z = K / G$, where $K$ is the bulk modulus.
The resultant plastic stiffness tensor has the following form:

\[ C_{ijkl}^p = \frac{G}{H} (M_{ik}^{(s)} + z\delta_{ik})(M_{jl}^{(s)} + z\mu\delta_{jl}) \]  

(Equation 17c)

**Evolution of friction and dilatancy**

The mobilized friction coefficient \( \mu \) (Equation (13)) is assumed to be a function of the effective plastic strain as

\[ \mu(\gamma^p) = \mu_{cv} + x_1 \left( \frac{\gamma^p + \mu_0 - \mu_{cv}}{x_1} \right) \exp(-x_2\gamma^p) \]  

where \( \mu_{cv} \) is the internal friction at constant volume, \( \mu_0 \) is the initial mobilized friction, and \( x_1 \), \( x_2 \) are parameters determined by calibration from experimental results. Similar formulations relating \( \mu \) to \( \mu_{cv} \) have been adopted by other researchers [9, 20].

Following the work of Taylor [21] the dilatancy is expressed as (see also Reference [18]):

\[ \beta(\gamma^p) = \mu(\gamma^p) - \mu_{cv} \]  

The multi-slip gradient plasticity model has been implemented into the FE commercial code ABAQUS [22] as a special user material (UMAT) subroutine for the case of two active slip systems. It has been used to study the characteristics of strain localization and shear band initiation in a variety of situations. A complete description of such problems has been presented in Reference [10]. The model is used here to identify the conditions that lead to the observed stress distributions beneath the granular heap.

**STRESS DISTRIBUTION BENEATH A CONICAL GRANULAR PILE**

As highlighted before several researchers have performed detailed experiments to measure the vertical stress distribution at the base of granular piles with widely varying observations. This study models the experiments reported by Vanel *et al.* [4] as their series of experiments were one of the few where an attempt was made to compare the effects of construction procedure on the stress distribution.

**Experimental setup**

Vanel *et al.* [4] conducted a serial of experiment with two different construction techniques to explore the stress dip underneath the sand pile. The sand piles were made using particles with diameter 1.2 mm ± 0.4 mm and angle of repose 33°. The piles were constructed on a Duralumin base plate 15.0 mm thick, adequate enough to prevent deflection under the weight of the pile. A single capacitive pressure sensor of diameter of 11.3 mm (≈ 9 grain diameters) was placed flush with the surface of the base plate. The normal stresses at various locations were determined along the radial axis of the conical piles by repeated construction of heaps with the same mass of sand. Two types of heaps were constructed by using distinct procedures. The ‘localized source’ procedure used a funnel; while the homogenous deposition or ‘raining procedure’ used a sieve (Figure 4). The dimensionless vertical normal stress \( (\sigma_{22}/\rho g H) \) for the two procedures is also shown in Figure 4 with the evidence of dip in the case of localized source and no evidence of dip in the case of raining procedure.
Numerical simulation

The stress distributions within the sand pile were determined using finite element analysis for various conditions. The conical pile was discretized using four-node bilinear axisymmetric quadrilateral reduced integration solid element (CAX4R) from the ABAQUS library. Due to axial symmetry, only half of the problem needed to be studied as shown in Figure 5. The geometry was chosen to have an angle of repose of 33.0°. Since the angle of repose represents the angle of internal friction of the granular materials at its loosest state, it is assumed here to be equal to the angle of internal friction at constant volume or equivalently, the constant volume friction ($\mu_{cv}$) = 0.649. In addition, the following parameters were used in the gradient double slip model: Initial mobilized friction ($\mu_0$) = 0.00; $E$ = 200 MPa, Poisson ratio, $v = 0.20$, $x_1 = 3.00$, $x_2 = 1055.00$ and $c_1 = 0.00$.

Figure 4. Conical piles of granular materials (height $H$ and radius $R$). Dimensionless normal stress profiles, $P = (\sigma_{zz})/(\rho g H)$, vs dimensionless radial distance $r/R$: (a) construction technique; (i) local source method (ii) raining (homogeneous) method (after Reference [4]). (Reprinted Figure 2 with permission from Memories in sand: Experimental tests of construction history on stress distributions under sandpiles, L. Vanel, D. Howell, D. Clark, R. P. Behringer and E. Clément, Phys. Rev. E, 60:R5040–R5043, 1999. DOI:10.1103/PhysRevE.60.R5040 Copyright © 1999 American Physical Society.)

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The model was subjected only to gravitational loads in the vertical direction, and rough friction contact was imposed along the rigid base of the sand pile. The construction of the granular heap was simulated in five stages as shown in Figure 6. The results of the normalized vertical stress distribution along the base for given slip system are shown in Figure 7.

Figure 5. FE model configuration.

Figure 6. Sequential staged loading for granular heap.
It is necessary firstly to choose a set of initial active planes in the double slip model. As mentioned before, in the case of metals these are well defined. But in the case of granular materials they are not and depend on the microstructure and history of deposition. For homogenous state of granular materials, following the classical Mohr–Coulomb solution we first assume that shearing is possible only on those initial slip systems with \( \zeta_1 = \pi/4 + \phi/2 \) and \( \zeta_2 = -\pi/4 - \phi/2 \), where \( \phi \) is the angle of repose. The normalized vertical stress distribution along the base of the sand pile determined using this set is as shown in Figure 8. It is evident that

![Diagram](image)

Figure 7. Progress of normalized stress distribution during sequential staged loading, \( \zeta_1 = \pi/4 + \phi, \zeta_2 = -\pi/4 - \phi \).

![Diagram](image)

Figure 8. Normalized vertical stress distribution at the base of sand pile; initial slip system orientation at \( \zeta_1 = \pi/4 + \phi/2, \zeta_2 = -\pi/4 - \phi/2 \).

for this case, the stress profile increases monotonically with the peak occurring beneath the
apex. Therefore, for sand piles with initial slip systems $\zeta_1 = \pi/4 + \phi/2$ and $\zeta_2 = -\pi/4 - \phi/2$ the stress dip as observed by some past investigators does not arise. Since sand piles produced using a raining procedure would result in a homogeneous state that satisfies the initial slip systems with $\zeta_1 = \pi/4 + \phi/2$, $\zeta_2 = -\pi/4 - \phi/2$, as observed by some investigators [4], such procedures would not result in a stress dip.

On the other hand, if the initial slip system is controlled by the microstructure and history of
deposition, their orientation will necessarily differ from $\zeta_1 = \pi/4 + \phi/2$ and $\zeta_2 = -\pi/4 - \phi/2$. This is especially true in the case of localized source of deposition methods. In order to simulate such problems, the slip systems were varied using $\zeta_1 = (\pi/4 + \phi/2) + \Gamma$ and $\zeta_2 = (-\pi/4 - \phi/2) - \Gamma$, where $\Gamma$ is a constant that was varied from 0.0 to 3.0 times $\phi$.

The effect of initial slip system orientation on the observed vertical stress distribution along
the base of the sand pile is as shown in Figure 9. It can be seen that regardless of the initial slip
orientation the stress distribution increases monotonically up to a ring of normalized radius ratio around 0.2. However, the stress distributions other than that for the homogenous state ($\zeta_1 = \pi/4 + \phi/2$, and $\zeta_2 = -\pi/4 - \phi/2$) begin to decrease and reach a minimum at the apex. The amount of decrease or the dip is dependent on the initial microstructure and thus the initial slip orientation. The maximum amount of drop achieved was slightly over 50% from the peak observed at the outer edges of the ring. Therefore, we believe that the initial microstructure resulting from sand deposition would result in the development of different initial active slip systems and lead to differences in stress profile within the ring. Consequently, the classical Mohr–
Coulomb solutions cannot be used to calculate such stress distributions.

![Figure 9. Normalized vertical stress distribution at the base of sand pile for different initial slip systems.](attachment:image.png)
The contour plots for vertical stress and vertical elastic and plastic strain within the sand pile reveal patterns that confirm the trend in the vertical stress distribution observed along the base (Figures 10 and 11). The figures are shown for two cases; one with slip system orientation

![Contour plots](image)

**Figure 10.** Contour plots of the vertical stress distribution ($\sigma_{22}$) for two different initial slip systems: (a) $\zeta_1 = \pi/4 + \phi/2$, $\zeta_2 = -\pi/4 - \phi/2$; and (b) $\zeta_1 = \pi/4 + \phi$, $\zeta_2 = -\pi/4 - \phi$.

![Contour plots](image)

**Figure 11.** Contour plots of the vertical elastic and plastic strain for two different initial slip systems, 856 elements: (a) $\zeta_1 = \pi/4 + \phi/2$, $\zeta_2 = -\pi/4 - \phi/2$; and (b) $\zeta_1 = \pi/4 + \phi$, $\zeta_2 = -\pi/4 - \phi$. 

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\( \zeta_1 = \pi/4 + \phi/2 \) and \( \zeta_2 = -\pi/4 - \phi/2 \) that showed no dip in stress and the second with \( \zeta_1 = \pi/4 + \phi \) and \( \zeta_2 = -\pi/4 - \phi \) that showed the maximum dip. It can be seen that beyond the ring of normalized radius of 0.2 the stress and strain pattern within the pile is similar for both

![Image of normalized vertical stress distribution at the base of sand pile.](image1)

**Figure 12.** Density influence on normalized vertical stress distribution at the base of sand pile.

![Image of normalized vertical stress distribution beneath the sand pile.](image2)

**Figure 13.** Normalized vertical stress distribution beneath the sand pile for same initial slip system orientation and different mesh refinement.
cases, with plastic deformation confined within localized region around the apex while the rest of the pile is in an elastic state of deformation. However, within the ring the contours differ substantially confirming the effects of localization.

The effect of density of the sand on one of the stress distribution that showed the maximum dip is as shown in Figure 12. The results show that the density is not a factor in the dip or the stress profile. Since this phenomenon is the result of localization, analyses were conducted to study the influence of FE mesh sensitivity and the results are as shown in Figure 13. The results show small variation with mesh size and this can be accounted for by a using a gradient term in the double slip formulation (see Reference [19]).

CONCLUSIONS

A multi-slip gradient formulation using the pressure dependent plastic yield surface is presented in this paper to model strain localization in granular materials. The model has been incorporated into a finite element code and used to study the stress distribution at the base of a granular pile. Based on the study the following conclusions can be drawn:

(i) The vertical stress distribution along the base of granular pile is dependent on the initial slip orientation.
(ii) The stress distribution increases monotonically up to the apex of the pile for homogeneous state of granular materials with initial slip orientation ($\zeta_1 = \pi/4 + \phi/2$ and $\zeta_2 = -\pi/4 - \phi/2$).
(iii) The initial slip system is dependent on the microstructure and history of deposition of sand.
(iv) The stress distribution for sands with initial slip orientation different from $\zeta_1 = \pi/4 + \phi/2$ and $\zeta_2 = -\pi/4 - \phi/2$ results in the peak of the stress achieved at a ring of normalized radius at around 0.2. Thereafter, the stress dips reaches minimum value beneath the apex. The extent of the dip is dependent on the orientation of the slip system.
(v) The extent of stress dip is not dependent on the density of the sand.

The study has demonstrated that the use of double slip formulation is effective in capturing the effects of localization on the vertical stress distribution in granular heaps, which cannot be accounted for by classical continuum formulations.

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