EFFECT OF INITIAL FABRIC TENSOR ON LOCALIZATION OF DRY SAND

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ABSTRACT:
Localisation of deformation into narrow shear bands is a fundamental phenomenon in granular materials. Constitutive equations that do not consider the effect of higher order deformation gradients fail to predict deformation patterns that persist and eventually dominate shear bands. This paper presents a higher order gradient multi-slip formulation to model the effect of inhomogeneous deformation in granular materials. The effects of inherent anisotropy within the multi-slip formulation are accounted for directly. The mobilized friction in a pressure dependent yield surface is assumed to be a direct function of the fabric tensor. The formulation with two active slip systems has been implemented into a finite element code and used to simulate biaxial shear tests on dry sand. The analysis is quantifies most of the shear band characteristics observed by past experimentation. It is found that initial fabric lead to very different localised behaviour in granular materials.

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KEYWORDS: Dilatancy, Frictional Materials, Elastic/Plastic, Fabric Anisotropy, Shear Band, Slip, Gradient Deformation.

INTRODUCTION
Constitutive equations for elasto-plastic deformation using the first deformation gradient and its history at that point are valid for simple materials in the sense of Noll (1958). The dimension of the underlying microstructure becomes relevant when deformation is heterogeneous and local strain gradients become intense, consequently, altering the material behaviour. Therefore, when deformation fields are highly heterogeneous, constitutive equations that do not consider the effect of higher order gradients fail to predict deformation patterns that may persist and eventually dominate deformation mechanisms such as shear bands.
The shear band phenomenon in granular materials has been studied extensively (e.g. Roscoe 1970; Zbib and Aifantis, 1989; de Borst and Muhlhaus, 1992; Vardoulakis, 1996; Tejchman and Gudehus, 2001; Desrues and Viggiani 2004). Some of these studies have introduced various higher gradient measures to incorporate microstructure length scale into the classical formulations and have used them to study strain localization and shear bands. Most of them, however, are phenomenological in nature. Therefore, there is a need to relate them with the underlying granular microstructure.

The mechanical behaviour of soils is strongly influenced by the packing of individual particles. Porosity or the void ratio is often used to characterize the state of packing. These scalar measures, however, are insufficient to characterize the directional behaviour of granular materials. Therefore, higher order microstructural variables known as “fabric tensors” has been used to describe the distribution and orientation of grains and voids (Oda et al., 1982; Tobita, 1989; Bathurst and Rothenburg, 1990; Pietruszczak and Krucinski (1989a); Muhunthan et al. 1996). Attention in such studies has been focused on the granular material behaviour in the strain-hardening regime far from the region where plastic instabilities occur. On the other hand, it has been pointed out that higher order microstructural features and related continuum models of structured media are particularly important near instabilities such as in the strain-softening regime (Vardoulakis and Aifantis, 1991).

Experiments on various assemblies of discrete particles have shown that the overall deformation of a granular mass consists of simple dilatant shearing deformations on a number of active shearing planes (Oda et al. 1982, 1985; Nemat Nasser 2000). The sliding planes are similar to the slip planes in crystal plasticity. Therefore, crystal plasticity type formulations can be used to develop a micromechanics based model for granular materials. “Double-shearing” plane strain constitutive model for granular materials have been used in the past by Mehrabadi & Cowin (1978), Anand (1983), and Zbib (1991&93). Recently, Anand and Gu (2000) have included dilatancy in their double slip model and implemented it in the finite element code ABAQUS. None of these studies, however, incorporate higher gradient measures of length and therefore are ineffective to capture strain localization.

This paper presents a second order gradient multi-slip plasticity formulation to study the localization phenomenon in granular materials. The effects of anisotropy within the multi-slip formulation are accounted for directly by assuming the mobilized friction in a pressure dependent yield surface formulation to be a function of the fabric tensor. A two active slip version of the multi slip formulation has been implemented into ABAQUS (2003) and used to study shear bands and post localization behaviour in granular materials.

**Constitutive Equations**

**Mathematical Preliminaries**

The velocity gradient ($L_{ij}$) of a continuous medium undergoing a smooth deformation can be split into two parts; symmetric and skew-symmetric. The symmetric part represents the strain rate tensor, $D_{ij}$, and the skew-symmetric part represents the spin tensor, $W_{ij}$:

$$D_{ij} = \frac{1}{2} \left( L_{ij} + L_{ji}^{T} \right), \quad W_{ij} = \frac{1}{2} \left( L_{ij} - L_{ji}^{T} \right)$$

(1)

The strain rate tensor, $D_{ij}$, and the spin tensor, $W_{ij}$, can be decomposed as:

$$D_{ij} = D_{ij}^{s} + D_{ij}^{p}, \quad W_{ij} = \omega_{ij} + W_{ij}^{p}$$

(2)

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where , , and $D_{ij}^e D_{ij}^p W_{ij}^f \alpha_{ij}$, are the elastic, plastic parts, plastic spin and the spin of microstructure, respectively.

$D_{ij}$ can be, also, split into a volumetric strain rate ($D$), and a deviatoric strain rate $d_{ij}$ as:

$$
D = D_{\text{vol}}^e = D_{ij} - \frac{1}{3} \delta_{ij} D_{kk}
$$

where $\delta_{ij}$ is the Kronecker delta.

The equation of equilibrium for quasi-static loading conditions is given by:

$$
\sigma_{ij} + b_i = 0
$$

where $\sigma_{ij}$ is the Cauchy stress tensor and $b_i$ is the body force per unit volume.

The stress changes due to the deformation of granular body is assumed to follow Hooke’s law:

$$
\sigma_{ij} = C_{ijkl} D_{kl}^e
$$

$C_{ijkl}$ is the elasticity tensor, $G$ the shear modulus, and $\nu$ the Poisson's ratio. The Jaumann objective stress tensor is defined with respect to the frame rotating with the material filament by

$$
\sigma_{ij} = \dot{\sigma}_{ij} - \omega_{k} \sigma_{kj} + \sigma_{kj} \omega_{i}
$$

**Multi - Slip Model For Plastic Flow**

In the multi-slip model, the plastic deformation of the granular materials is viewed in terms of shearing and dilation of the grains assembly on two planes defined by their normal unit vector and slip direction. The approach adopted here is a generalization of the “double-shearing” plane strain (DSPS) constitutive model (Spencer (1964), Mehrabadi & Cowin (1978), Anand (1983), Zbib (1993) & Spencer (2003)). The DSPS is generalized to three dimensions to include the effects of elastic deformation, pressure dependent, shearing dilation, and hardening/softening response observed in granular materials, as well as the effect of heterogeneous microstructure.

The model is extended to include a gradient term of the effective plastic strain within a set of yield functions to preserve the elliptical type of the governing equation in static loading. This also, enables to identify the dimension of the internal length scale of the granular materials under investigation. The length scale becomes relevant when deformation is heterogeneous and local strain gradients become intense, consequently, altering the material behavior (Al Hattamleh et al. 2004).

The orientations of the slip planes are assumed to depend on the maximum shear stress obliquity and on the deposition angle of granular materials particles. In addition, the rotation of the resolved slip plane and the orthogonal plane are accounted for by updating the values
of unit vectors \((m^{(s)}_i = (m^{(s)}_{ij}, n^{(s)}_{ij}) = \frac{2N^{(s)}_i}{m^{(s)}_{ij}} \cdot n_i)\) as:

\[
\hat{m}_i = \omega_\theta m_j; \hat{n}_i = \omega_\theta n_j \quad (7)
\]

The components of one of the slip systems \(s(1)\) is given by:

\[
m^{(s)}_j = -\cos \zeta_j \cdot e_j - \sin \zeta_j \cdot e_z; \quad (8) \quad n^{(s)}_j = \sin \zeta_j \cdot e_j - \cos \zeta_j \cdot e_z;
\]

where \(\zeta_j\) is the angle measured with respect to the minor principal stress axis and \(e_1, e_2\) are the unit vectors in the Cartesian coordinate system.

**Yield and Plastic Potential functions**

The set of yield functions on the active slip system of the granular material is assumed to follow the pressure dependent yield surface, \(f^{(s)}\):

\[
f^{(s)} = q^{(s)} - \mu \sigma^{(s)} - l^2 G\nabla^2 f^{(s)} \quad (9)
\]

where, \(q^{(s)}\) and \(\sigma^{(s)}\) are the resolved shear and normal stresses, \(\mu\) is the mobilized friction coefficient, \(l\) represents the material length scale and \(G\) represents the Laplacian of effective plastic strain, \(\nabla^2 f^{(s)} \quad (9)
\]

The resolved shear and normal stresses can be expressed as functions of Cauchy stress as:

\[
\sigma^{(s)} = N^{(s)}_y; \quad (10) \quad \sigma^{(s)} = N^{(s)}_y;
\]

The deviatoric plastic strain rate is defined by means of a set of potential plastic function \(Q_{\psi} \psi^{(s)}\) as:

\[
d^{(s)} = \frac{\partial Q^{(s)}}{\partial \sigma} \quad (11)
\]

wherein \(d^{(s)} = \sqrt{2} \frac{d^{(s)}_x d^{(s)}_y}{}\) is the effective plastic strain rate.

The plastic potential function \(Q^{(s)}\) is assumed in this model to be

\[
Q^{(s)} = q^{(s)} + \beta \psi^{(s)}; \quad \psi^{(s)} = \int \psi^{(s)} dt \quad (12)
\]

where \(\beta\) is the mobilized dilatancy coefficient which relate volumetric plastic strain to effective plastic shear strain as:

\[
D^{(s)} = \beta \sum_1^n \phi^{(s)} \quad (13)
\]

Substituting Eq. (12) into Eq. (11) and combining with Eq. (13), leads to the plastic strain rate tensor in the form

\[
D^{(s)}_i = \sum_1^n \dot{\gamma}^{(s)}_i (M^{(s)}_{ij} + \beta N^{(s)}_{ij}) \quad (14)
\]

Moreover, the plastic spin can be found as (Al Hattamleh et al., 2004):

\[
W^{(s)}_i = \sum_1^n \dot{\gamma}^{(s)}_i (m^{(s)}_{ij} n^{(s)}_{ij} - n^{(s)}_{ij} m^{(s)}_{ij}) / 2 \quad (15)
\]
Effective plastic shear strain rate and stiffness tensors

Utilizing the yield and consistency conditions, along with Eqs. (5), (9), (10) and (15), the effective plastic strain rate can be evaluated as (Al Hattamleh 2006):

\[
\dot{\gamma} = \frac{(M_u^{(s)} + \alpha \mu \delta_\uparrow) : D_u}{1.0 + |p^{(s)} h_t / G + \alpha \beta \mu + l^2 \gamma^{(s)}|}
\]  

where \( h_t \) = \( \delta \mu / \delta \gamma^{(s)} \) dl / d\( \gamma^{(s)} \alpha \) = K / G, and K is the bulk modulus.

Substituting Eq. (16) and Eq. (14) into Eq. (2a) and combining it with Eqs. (5) and letting \( \dot{\gamma} = 0 \),

\[
\sigma_{ij}^e = (C_{ijkl}^{e} - C_{ijkl}^{p}) D_{kl} - \sum_{s=1}^{2} l^2 G V^2 \gamma^{(s)} (M_u^{(s)} + \alpha \beta \delta_\uparrow) H^{(s)}
\]  

The resultant plastic stiffness tensor is given as:

\[
C_{ijkl}^{p} = \frac{G}{H^{(s)}} (M_u^{(s)} + \alpha \beta \delta_\uparrow) (M_u^{(s)} + \alpha \mu \delta_\uparrow)
\]  

Moreover, \( l \) is as in Al Hattamleh et al. 2004

\[
l = \left[ \frac{D_{50}}{4} \right]^{1/3} \left[ 1 + \left( \frac{|p^{(s)} h_t / G + \alpha \beta \mu|}{\gamma^{(s)}} \right) \right]^{1/2} \approx \left[ \frac{D_{50}}{4} \right]^{1/2} \]  

where \( D_{50} \) is main grain diameter of soil.

Evolution of Friction and Dilatancy

The mobilized friction coefficient \( \mu \) (Eq. 9) is assumed to be a function of the effective plastic strain:

\[
\mu(\gamma^{(s)}) = \mu_s + x_s (\gamma^{(s)} - \mu_s e^{-s^{(s)}})
\]  

where \( \mu_s \) is assumed initial to be equal to \( \mu_{cs} \), which is the internal friction at constant volume, and \( s \), are material parameters determined by trial and error from experimental results.

Similar formulations relating \( \mu \) to \( x_s \mu_{cs} \) have been used in the past (Balendran and Nemat Nasser 1993; Anand and Gu 2000, Al Hattamleh et al. (2006)). It is noted that the mobilized friction is affected by porosity heterogeneity and fabric anisotropy. The formulation above is modified later to account for these phenomena.

Following the work of Taylor (1948) the dilatancy is expressed as (Vardoulakis, 1996):

\[
\beta(\gamma^{(s)}) = \mu(\gamma^{(s)}) - \mu_{cs}
\]  

Plane Strain Experiments and Model Configuration

The multi slip gradient plasticity formulation has been implemented into the ABAQUS (2003) standard Finite Element code as a special user material UMAT subroutine for the case of two-active slip systems. The multi slip gradient model was used to study the characteristics of strain localization and shear band initiation.
Han and Drescher (1993) have reported some biaxial shear tests on dry sand. The biaxial apparatus used a prismatic specimen of 40 mm width, 80 mm length, and 140 mm height. The specimen was enclosed between two rigid walls 80 mm apart and placed on a platen, which rested on a linear bearing. The linear bearing provided kinematic freedom for the formation of shear bands with the lower specimen portion sliding horizontally. The apparatus was placed inside a pressure chamber and the specimen was subjected to a confining pressure and kinematically or statically controlled axial load.

Biaxial tests were performed on coarse, poorly graded Ottawa sand with rounded particles of mean grain diameter $D_{50} = 0.72$ mm. Homogenous dense specimens, with an initial porosity of $\xi_0 = 0.32-0.33$, were prepared. All tests were performed with displacement controlled axial loading. Additional details of the test device and measurements of relevant parameters are provided in Han and Drescher (1993). The data for mobilized friction reported by Han and Drescher (1993) is used here to fit the proposed evolution equation (Eq. 20).

Numerical Simulation of Biaxial Tests

The plane strain has been modelled using a plane strain finite element analysis. All analyses were carried out using four-node plane strain elements with reduced integration, CPE4R. The total number of mesh elements was varied in order to study the effect of strain localization and shear band on mesh sensitivity. The lower boundary of the mesh is assumed to be rigid.

In the first phase, a confining pressure was applied to consolidate the specimen isotropically. Thereafter, axial compression was applied by increasing the vertical displacement on the top of the specimen. Based on the characteristics of the sand and the test conditions, the following material constants were used in the gradient multi-slip model: Stiffness modulus (E) = 180 MN/m$^2$; Poisson ratio (\(\nu\)) = 0.2; Constant Volume friction (\(\mu_{cv}\)) = 0.61; \(x_1 = 25.0; x_2 = 55.0; \xi_{crit} = 0.43\). The multi slip model with two active slip systems along and \(\zeta_1 = 60.75\) and \(\zeta_2 = -60.75\) was used.

The complete force displacement curve as predicted by the FE analysis is shown in Fig. 1. It was found that the force displacement prediction is dependent on the size of mesh refinement. However, it had also been found that the analysis fits the experimental data well for mesh size of 20 x 40. Therefore, no further refinement is required beyond this size.

Shear band formation is known to be influenced by confining pressure. Han and Drescher (1993) reported decreases of shear band angle, measured with respect to minor principal stress direction, as confining pressure increases. The effect of confining pressure on shear band initiation from the FE analysis is as shown in Fig. 2 for a 10 x 35 mesh. It can be seen that at lower confining pressures the appearance of shear band is evident. Furthermore, it appears that the inclination of the shear band reduces with increasing confining pressure just as observed by previous experiments. However, at higher confining pressures the localization of deformation into shear band appears to be non existent (Fig. 2). Similar observations are evident from Figure 3 which shows the results for a finer 20 x 40 mesh.
The inclination of the resulting shear band with the introduction of weak elements in the central portion of the mesh was measured to be $56\,^\circ \pm 2\,^\circ$ relative to the horizontal axis depending on the number of total meshes (Fig. 2 & 3). The experimentally measured angle was found to vary between $55\,^\circ$ to $56\,^\circ$ for confining pressure of 200 kPa (Han and Drescher 1993). This prediction lies between the suggestion value of shear band inclination by Roscoe (1970) $\Theta = \pi/4 + \psi/2$ and the $\Theta = \pi/4 + (\phi + \psi)/4$ which reported by Arthur et al. (1977) experimentally and proven theoretically by Vardoulakis (1980), where $\psi$ is the dilation angle measured at onset of localization.

Effect of fabric Anisotropy on mobilized friction: Void Fabric Measure

The directional distribution of the mobilized friction within a Representative Elemental Area REA can be approximated similar to void fabric tensor given by (Kanatani, 1984; Pietruszczak and Krucinski (1989a, b); Muhunthan et al. 1996) and as Pietruszczak and Pande (2001) to be:

$$\mu(\gamma^p, \mu, \Theta_{ij}) = \mu_m \left( I + \Theta_{ij} n_i n_j \right)$$  \hspace{1cm} (21)

where, the components of the unit vector $n_i$ are given in Eq.8. The mean mobilized friction, $\mu_m$ is assumed to have the same form as Eq.20, while the mobilized friction fabric tensor, $\Theta_{ij}$, is given by:

$$\Theta_{ij} = \frac{15}{2} \left( \psi_{ij} - \frac{1}{3} \delta_{ij} \right)$$ \hspace{1cm} (22)

with the second moment tensor $\psi_{ij}$ (Kanatani 1984):

$$\psi_{ij} = \frac{1}{4\pi \mu_m} \int n_i n_j \mu(n) dn$$ \hspace{1cm} (23)

Note that when the components of the mobilized friction fabric tensors are zero, Eq. (21) reduces to the isotropic form with mean mobilized friction as used in the previous section of the current studies. Hence, the mobilized friction fabric tensor reflects the deviations from isotropy. Moreover the dilatancy in Eq. 21 was modified to have the form of:

$$\beta(\gamma^p, \mu, \Theta_{ij}) = \mu(\gamma^p, \mu, \Theta_{ij}) - \mu_m \left( I + \Theta_{ij} n_i n_j \right)$$ \hspace{1cm} (27)

Fabric effects on shear band and Mobilized Friction

In order to examine the effects of fabric on such shear band characteristics it was decided to use some data on photo elastic measurements of initial fabric for samples reported by Oda et al. (1985) in biaxial shear tests. The specimens have an oval shaped particles deposited at different angles. The corresponding initial porosity fabric tensor values $\Omega_{ij}$ was determined by Muhunthan (1991) for the different cases as shown in Table 1. Assuming that all other mechanical properties remain the same and initial porosity fabric tensor has same values as mobilized friction fabric tensor, these values can be used to study the influence of initial fabric on localization for the biaxial tests used here.
The biaxial tests were simulated using 350 elements of a four node plane strain element with reduced integration, CPE4R mesh, subjected to constant lateral pressure, and axial displacement level to 10% of the specimen height. The two activated slip systems are $\gamma_1 = 60.75$ and $\gamma_2 = -60.75$.

Variation of the mobilized and mean mobilized friction with deposition angle is shown in Figure 4. It can be shown that the deposition angle values of 60° gives the lowest mobilized friction curve which in turn to be the lowest of peak and residual friction angle as shown in Figure 5. Similar trend are reported by Pietruszczak and Pande (2001) through their analysis of plane strain condition of normally consolidated soil sample using the multi-laminate framework concept.

The deformed shapes for the different deposit angle cases without and with using the fabric based dilatancy Eq.24 are as shown in Fig. 6 and Fig. 7 respectively. It can be seen that the localization of deformation into shear band is dependent very much on the depositional angle, and on the use of fabric based dilatancy Equation. The use fabric tensor will enable the quantification of such shear bands and their orientation in granular materials in the future.

CONCLUSIONS

A multi-slip second gradient formulation is presented in this paper to model strain localization in dry sand. Double slip version of the model has been incorporated into a finite element code and used to simulate plane strain test results on dry sand. The effects of granular microstructure have been accounted for by assuming the mobilized friction to be a function of fabric tensor measure. Deformation found to localize into a narrow shear band depend on the level of confining pressure, and inclination of shear band angle obtain between $\Theta = \pi/4 + \psi/2$ and $\Theta = \pi/4 + (\phi + \psi)/4$. Moreover, Strain localization and shear band characteristics are dependent very much on the deposition angle of granular materials, which in turn affects the values of fabric tensor components.

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REFERENCES


**Table 1. Initial void fabric tensor measurements on plane strain test (Muhunthan 1991)**

<table>
<thead>
<tr>
<th>deposition angle*, $\theta_d$</th>
<th>State</th>
<th>$\Omega_{11}$</th>
<th>$\Omega_{12}$</th>
<th>$\Omega_{22}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>Initial</td>
<td>-0.04</td>
<td>0.07</td>
<td>0.04</td>
</tr>
<tr>
<td>60°</td>
<td>Initial</td>
<td>0.86</td>
<td>0.16</td>
<td>-0.86</td>
</tr>
<tr>
<td>90°</td>
<td>Initial</td>
<td>0.54</td>
<td>0.09</td>
<td>-0.54</td>
</tr>
</tbody>
</table>

* Deposited with respect to the horizontal plane
Fig. 1: Force displacement curve for $\sigma_1 = 60.75$ and $\sigma_2 = -60.75$

Fig. 2: Deformed shape at different confining pressure levels for a 10 x 35 mesh ($\sigma$ = confining pressure, $\sigma_1 = 60.75$ and $\sigma_2 = -60.75$)
Fig. 3: Deformed shape at different confining pressure levels for a 20 x 40 mesh ($\sigma_c$ = confining pressure, $\zeta_1 = 60.75$ and $\zeta_2 = -60.75$)

Fig. 4: Variation of the Mobilized friction with deposition angle, $\theta$ (Note $\theta = N/A$ corresponded to the mean mobilized friction)
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Fig. 5: Variation of the Peak and Residual friction angle with deposition angle

Fig. 6: Deformation shape for a mesh with 10 x 35 elements for ideal oval particle shapes with out using the fabric based dilatancy equation, ($\theta$ = deposition angle)
Fig. 7: Deformation shape for a mesh with 10 x 35 elements for ideal oval particle shapes with using the fabric based dilatancy equation, ($\theta =$ deposition angle)