Behavior of a rhombus frame of nonlinear elastic material under large deflection

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1. Introduction

Highly flexible elements are widely used in precision mechanics, especially in aerospace products and biomedical instrumentations, where high strain and fatigue capabilities are required. Examples of such elements include flexural pivots, parallel spring translators, flexible linking devices, nonlinear mechanical springs, compliant gripper elements and microelectromechanical (MEMS) devices. The increasing use of such elements requires a better understanding of their behavior under large deflection.

Most of the applications of nonlinear bending theory to thin beams have been confined to beams of linear elastic materials [1–16]. Fewer studies are available for large deflections of beams made of a nonlinear elastic material. Oden and Childs [17] studied the nonlinear post-buckling behavior of a thin prismatic cantilever beam. The nonlinear elastic material was modeled using a moment–curvature relationship in the form of hyperbolic tangent law. Parathep and Varadan [18] investigated the inelastic large deflection of a prismatic cantilever beam with a rectangular cross-section subjected to a vertical point load applied at the free end.

The material of the beam was considered to have a stress–strain law of the Ramberg–Osgood type. Monasa [19] studied the effect of material nonlinearity on the stability behavior of a thin prismatic cantilever column. The material of the column was represented by a logarithmic stress–strain function. Lewis and Monasa [20] solved the large-deflection problem of a slender prismatic cantilever beam subjected to a tip vertical load using a numerical integration technique. The stress–strain relationship of the nonlinear material used was of a Ludwick type.

Furthermore, Yu and Johnson [21] investigated the post-buckling elastic–plastic deformation of a strut. The researchers extended the theory of elastica [22] to determine the shape of a strut undergoing large plastic deflection. The equations governing such a behavior, known as the plastica [21] equations, were set up and solved by a perturbation method and by numerical integration. Lewis et al. [23] investigated the large-deflection and stability behavior of a vertical slender column of elastic–plastic material subjected to uniformly distributed loads, including its own weight, using the variational approach. Luan and Yu [24] studied the elastic–plastic large-deflection analysis of cantilever beams subjected to an inclined tip concentrated load. Fertis and Lee [25] and Fertis [26] studied the complicated nonlinear elastic problem of prismatic and nonprismatic inextensible beams using the method of equivalent systems.
Recently, Lee [27] solved the large-deflection problem of a slender prismatic cantilever of a Ludwick-type material subjected to a uniformly distributed load and a tip vertical load using a numerical integration technique. Huang et al. [28] considered the large-deflection behavior of elastic–plastic, nonlinear strain-hardening cantilevers of rectangular cross-sections, for which the stress–strain relationship after yielding is described by a Ludwick relation. Al-Sadder and Shatarat [29] investigated the large-deflection behavior of a prismatic composite cantilever beam made of two different nonlinear elastic materials and subjected to an inclined tip concentrated force.

The literature survey presented above shows that the large-deflection problem of linear elastic and nonlinear elastic beams has been extensively studied. However, published literature shows that the large-deflection problem of highly flexible frames has received little attention. Jenkins et al. [30] analyzed the large-deflection analysis of flexible portal type frames. Thacker et al. [31] investigated the behavior of a rectangular frame with rigid top and base, but with highly flexible prismatic side columns. The governing differential equations were integrated by homotopy and quasi-Newton methods. Wang [32] studied the nonlinear deformation of a loaded pinned–pinned triangular prismatic frame with an internal hinge at its apex. Ohtsukia and Ellyin [33] analyzed a square prismatic frame with rigid joints loaded diagonally. The analytical solution of the large-deflection problem was obtained using elliptic integrals. Dado and Al-Sadder [34] dealt with the elastic spring behavior of a rhombus frame consisting of nonprismatic members. The relations between the large displacement at the corners and the applied forces were obtained using a new robust numerical technique.

The rhombus frame has sometimes been referred to as a “half pantograph” in MEMS systems. Nielson and Howell [35] investigated the use of the pseudorigid body model (PRBM) to design fully compliant micro-half-pantographs. The PRBM was used to develop equations for input–output displacement, force–deflection and maximum nominal stress–deflection predictions.

To the authors’ knowledge, the large-deflection behavior of highly flexible frames having nonlinear elastic material has not been yet studied. This study deals with the large-deflection behavior of a rhombus frame having a solid rectangular cross-section with rigid joints. The rhombus frame is built up from a nonlinear Ludwick-type elastic material and is subjected to two opposite diagonal forces depicting a spring action. The differential equations governing the large-deflection behavior of the frame are formulated and solved. The solution is compared with the results obtained by the software ADINA [37]. The load–deflection curves, the deformed configurations and the nonlinear stiffness of the spring behavior are obtained for different nonlinear elastic material constants, \( n \), and apex angles, \( \gamma \).

### 2. Problem description

A slender inextensible rhombus frame with rigid joints having side lengths equal to \( 2L \) is subjected to a diagonal tension or compression force equal to \( 2P \) acting at point C, as shown in Fig. 1. Point A of the frame is held in position by a pin support. The side length of the rhombus makes an angle \( \beta \) with the \( x \)-axis, as shown in Fig. 1. The sides of the rhombus are rigidly jointed together so that the interior angle \( \gamma \) remains fixed regardless of the applied load or the deformation. Each side length of the rhombus frame has a solid rectangular cross-section and is made of a nonlinear Ludwick-type elastic material. The stress–strain relationship of such a material is of a power-law form [20] and can be expressed by a nonlinear function as follows:

\[
\sigma = B\varepsilon^n \quad (1)
\]

where \( \sigma \) and \( \varepsilon \) represent the stress and strain, respectively; \( B \) and \( n \) are constants that depend on the mechanical properties of the material.

### 3. Theoretical formulation

It is sufficient to investigate the behavior of a half-side member of the frame since the frame shown in Fig. 1 is doubly symmetric and the members of the frame are rigidly connected to each other. These two factors support the existence of an inflection point at the center of each side member. Therefore, a half-side member of the frame will be analyzed and the complete frame behavior will be constructed from the double symmetry. Figs. 2 and 3 show the half-side model of member \( AE \) for the rhombus frame under compression and tension in the Cartesian and rotated axes, respectively. The formulations of the governing differential equations are performed for the compression case in the rotated axes for simplicity and then generalized for the tension case.

An infinitesimal element of the beam having a length, \( ds \), is shown in Fig. 4. The element is under the action of three internal forces: a horizontal force, \( H \), a vertical force, \( V \), and an internal bending moment, \( M \). The coordinates of a point on this beam are denoted by \( x' \) and \( y' \). Angle \( \theta' \) is the slope of the beam measured from the positive \( x' \)-axis to the tangent of the corresponding point. The convective coordinate, \( s \), is the curved coordinate along the deformed length of the beam. Hence, six unknowns can be identified: \( H, V, M, x', y' \) and \( \theta' \). All these unknowns are functions of \( s \).
From Fig. 4, the geometrical relationships can be written as follows:

\[ \frac{dx_0}{ds} = \cos y_0 \]  
\[ \frac{dy_0}{ds} = \sin y_0 \]  

From Fig. 4, the equilibrium equations of the infinitesimal element are

\[ \frac{dV}{ds} = 0 \]  
\[ \frac{dH}{ds} = 0 \]  
\[ \frac{dM}{ds} = -V \cos \theta' + H \sin \theta' \]  

The sixth equation required to obtain a complete solution is the relationship between moment and curvature. This relationship can be obtained by providing an exact expression for the curvature by the use of Euler–Bernoulli moment–curvature relationship. Therefore, the moment–curvature relationship for a thin beam having a rectangular cross-section and made of a nonlinear elastic material described by Eq. (1) can be expressed as

\[ \frac{d\gamma}{ds} = \frac{M}{K} \]  

where \( K \) is the stiffness constant, which is dependent on the geometry of the cross-section and the material constant \( B \). It has the following form for solid rectangular sections of height \( h \) and breadth \( b \):

\[ K = \frac{2Bb(h/2)^{n+2}}{(n+2)} \]  

With reference to Eq. (8), the flexural stiffness of a beam made of a linear elastic material \((n = 1)\) is equal to \( K = \frac{6Bbh^3}{12} \), which is the well-known formula for the flexural stiffness of a solid rectangular cross-section.

3.1. Semi-analytical large-deflection solution for the rhombus frame

Eqs. (2)–(7) are six simultaneous nonlinear first-order differential equations. These nonlinear equations can be solved analytically as follows:

Integrating Eqs. (4) and (5) leads to

\[ V(s) = -P \cos \gamma \]  

Fig. 2. Rhombus frame subjected to compressive force: (a) loading configuration for half model in Cartesian and rotated coordinates and (b) deformed configuration for half model in Cartesian and rotated coordinates.

Fig. 3. Rhombus frame subjected to tensile force: (a) loading configuration for half model in Cartesian and rotated coordinates and (b) deformed configuration for half model in Cartesian and rotated coordinates.

Fig. 4. Internal forces acting on infinitesimal element \((ds)\) in \(xy'\)-coordinates in the deformed configuration state.
\[ H(s) = -P \sin \gamma \]  
where \( \gamma = \beta \), as shown in Fig. 2. Substituting Eqs. (9) and (10) into Eq. (6) gives
\[ \frac{dM}{ds} = P \cos(\gamma + \theta') \]  
Differentiating Eq. (7) once with respect to \( s \) gives
\[ n \left( \frac{d\theta'}{ds} \right)^{n-1} = \frac{1}{K} \frac{dM}{ds} \]  
Substituting Eq. (11) into Eq. (12) gives
\[ n \left( \frac{d\theta'}{ds} \right)^{n-1} = \frac{P}{K} \cos(\gamma + \theta') \]  
Multiplying both sides of Eq. (13) by \( d\theta'/ds \) it yields
\[ \frac{n}{n+1} \left( \frac{d\theta'}{ds} \right)^{n-1} = \frac{P}{K} \sin(\gamma + \theta') \]  
Integrating Eq. (15) once with respect to \( s \) gives
\[ \frac{n}{n+1} \left( \frac{d\theta'}{ds} \right)^{n-1} = \frac{P}{K} \sin(\gamma + \theta') + C_1 \]  
The constant of integration, \( C_1 \), can be obtained by applying the end condition at \( s = L \), where the bending moment is equal to zero, such that \( d\theta'/ds = 0 \), resulting in
\[ C_1 = -\frac{P}{K} \sin(\gamma + \theta_e) \]  
where \( \theta_e \) is the angle of rotation at the tip of the member AE as shown in Fig. 2. Substituting Eq. (17) into Eq. (16) gives
\[ \frac{n}{n+1} \left( \frac{d\theta'}{ds} \right)^{n-1} = \frac{P}{K} \sin(\gamma + \theta') - \sin(\gamma + \theta_e) \]  
or
\[ \frac{d\theta'}{ds} = \frac{P}{nK} \left( \frac{n+1}{n+1} \right)^{\frac{1}{n(n+1)}} \]  
In order to find the required force \( P \) for a given tip angle of rotation \( \theta_e \), Eq. (19) has to be applied at the free end of the cantilever (point E in Fig. 2) and the following expression is obtained:
\[ P = \frac{1}{L} \int_0^{\theta_e} \left( \frac{n+1}{nK} \sin(\gamma' + \theta') - \sin(\gamma' + \theta_e) \right) \frac{1}{(n+1)} d\theta' \]  
Now, after calculating the required force \( P \) for a given \( \theta_e \), the curved coordinate \( s \) must be obtained as a function of intermediate angle of rotation \( \theta' \) as follows. Recall Eq. (19):
\[ \frac{d\theta'}{ds} = \frac{P}{nK} \left( \frac{n+1}{n+1} \right)^{\frac{1}{n(n+1)}} \]  
or
\[ ds = \frac{n+1}{nK} \left( \frac{n+1}{nK} \sin(\gamma' + \theta') - \sin(\gamma' + \theta_e) \right)^{-\frac{1}{n+1}} d\theta' \]  
Integrating Eq. (22) once with respect to \( s \) gives
\[ s(\theta_e) = \int_0^{\theta_e} \frac{n+1}{nK} \left( \frac{n+1}{nK} \sin(\gamma' + \theta') - \sin(\gamma' + \theta_e) \right)^{-\frac{1}{n+1}} d\theta' \]  
Using Eqs. (2) and (19), an expression for the coordinate is obtained as follows:
\[ \chi(\theta_e) = \int_0^{\theta_e} \frac{1}{P^{\frac{1}{n+1}}} \cos \theta' d\theta' \]  
Similarly an expression for the coordinate \( y' \) is obtained:
\[ y'(\theta_e) = \int_0^{\theta_e} \frac{1}{P^{\frac{1}{n+1}}} \sin \theta' d\theta' \]  
The Cartesian coordinates \( x \) and \( y \) are obtained using simple transformation relations as follows:
\[ x(\theta_e) = \chi(\theta_e) \cos \beta - y'(\theta_e) \sin \beta \]  
\[ y(\theta_e) = \chi(\theta_e) \sin \beta + y'(\theta_e) \cos \beta \]  
The tip displacement at point E in Cartesian coordinates are given by
\[ x_e = \chi(\theta_e) \]  
\[ y_e = y(\theta_e) \]  
The bending moment \( M \) is obtained by substituting Eq. (19) into Eq. (7) so that
\[ M(\theta_e) = K e^{\frac{n+1}{nK}} \left( \frac{n+1}{nK} \sin(\gamma + \theta_e) - \sin(\gamma + \theta_e) \right)^{\frac{1}{n+1}} \]  
Eqs. (20), (23)–(25) and (30) apply for the case of applied compressive force. Similarly, one may obtain the relations for the case of applied tension force by letting the angle \( \gamma = \beta + \pi \) as shown in Fig. 3.

Further analysis was performed by applying numerical integration. Sixth-order Runge–Kutta algorithm was applied in order to integrate Eqs. (4), (5) and (9) using the following boundary conditions: \( x, y \) and \( \theta \) are equal to zero at \( s = 0 \).

4. Non-dimensional parameters

One good engineering application of this rhombus frame is to obtain a nonlinear stiffness behavior with favorable characteristics. Therefore, the following non-dimensional parameters are defined:

(1) The non-dimensional horizontal and vertical coordinates are
\[ x_n = \frac{x}{2L}; \quad y_n = \frac{y}{2L} \]  
(2) The non-dimensional load \( (2P) \) is
\[ x_n = \frac{(2P)2L^2}{K_e} \]  
where
\[ K_e = K|_{n=1} = \frac{\beta}{12} b^3 \]  
and \( K_e \) is the elastic secant stiffness in the direction of the load.

(3) The non-dimensional total deflection parameter in the direction of the load is
\[ \delta_c = \frac{4(y_e - L \sin \beta)}{2L} \]  
and the non-dimensional relative displacement of point B with respect to point D is
\[ \delta_{BD} = \frac{2x_e}{2L} \]  

Fig. 5. Load versus vertical deflection curves for different apex angles and $n$-values: (a) $n = 1.0$, (b) $n = 0.7$, (c) $n = 0.5$ and (d) $n = 0.3$ (compression case).

Fig. 6. Load versus vertical deflection curves for different apex angles and $n$-values: (a) $n = 1.0$, (b) $n = 0.7$, (c) $n = 0.5$ and (d) $n = 0.3$ (tension case).
Fig. 7. Load versus relative horizontal displacement, $d_{BD}$ curves, for different apex angles and $n$-values: (a) $n = 1.0$, (b) $n = 0.7$, (c) $n = 0.5$ and (d) $n = 0.3$ (compression case).

Fig. 8. Load versus relative horizontal displacement, $d_{BD}$ curves, for different apex angles and $n$-values: (a) $n = 1.0$, (b) $n = 0.7$, (c) $n = 0.5$ and (d) $n = 0.3$ (tension case).
Fig. 9. Nonlinear secant stiffness versus load parameter for different apex angles and $n$-values: (a) $n = 1.0$, (b) $n = 0.7$, (c) $n = 0.5$ and (d) $n = 0.3$ (compression case).

Fig. 10. Nonlinear secant stiffness versus load parameter for different apex angles and $n$-values: (a) $n = 1.0$, (b) $n = 0.7$, (c) $n = 0.5$ and (d) $n = 0.3$ (tension case).
Fig. 11. Deformed shapes of the rhombus under compression and for an apex angle of 15° with different $n$-values: (a) $n = 1.0$, (b) $n = 0.7$, (c) $n = 0.5$ and (d) $n = 0.3$.

Fig. 12. Deformed shapes of the rhombus under compression and for an apex angle of 45° with different $n$-values: (a) $n = 1.0$, (b) $n = 0.7$, (c) $n = 0.5$ and (d) $n = 0.3$. 
(4) The non-dimensional nonlinear secant stiffness in the direction of the load of the rhombus frame is

\[ K_{NL} = \frac{2a}{\delta_c} = \frac{(P/2L)^3}{K_E(y_E - L \sin \beta)} \]  

(35)

5. Large-displacement finite-element analysis using ADINA

To check the accuracy of the derived semi-analytical solution, a large-displacement finite-element analysis using the multi-purpose computer program ADINA was performed. Each side of the rhombus frame was divided into 10 moment-curvature beam elements. The following fundamental parameters were introduced to the program:

1. number of loading increments = 100,
2. maximum number of iterations during each increment = 25,
3. number of iterations before updating tangent stiffness matrix = 15,
4. method of updating tangent stiffness matrix—iterative,
5. tolerance for energy convergence criterion = \(1 \times 10^{-3}\),
6. solution strategy—full Newton-Raphson iteration method.

6. Numerical examples and discussion

Five different apex angles \(\gamma\) are considered for the rhombus frame in investigating the efficiency and the effectiveness of the derived analytical solution. For each of these frames, four values of the material constant, \(n\), are considered. Figs. 5 and 6 show the load versus vertical deflection curves for the five apex angles 75\(^\circ\), 60\(^\circ\), 45\(^\circ\), 30\(^\circ\) and 15\(^\circ\), and for compression and tension cases, respectively. Each figure has a plot for four different material constants, \(n\), 1.0, 0.7, 0.5 and 0.3. Figs. 7 and 8 show the load versus horizontal deflection curves for the five apex angles and for compression and tension cases, respectively. Each figure has a plot for four different material constants, \(n\), 1.0, 0.7, 0.5 and 0.3. For the same set of apex angles and material constants, the nonlinear secant stiffness \(K_{NL}\) is plotted against the non-dimensional relative horizontal displacement \(S_{DB}\), as shown in Figs. 9 and 10 for the compression and tension cases, respectively. Identical results were observed between the method presented in this study and results from ADINA. As could be predicted, the total elongation \(\delta_c\) behaves asymptotically as the load increases in tension to reach its limiting physical value given by

\[ \delta_c = 2 \left(1 - \cos \frac{\gamma}{2}\right) \]  

(39)

However in compression, the practical limiting value is when the load application point \(C\) coincides with the pinned corner \(A\), which may defined as

\[ \delta_c = -2 \cos \frac{\gamma}{2} \]  

(40)

The nonlinear stiffness behavior of the rhombus frame as depicted in Figs. 9 and 10 shows a hardening effect in tension and a softening effect in compression. However, in the tension case,
Fig. 14. Deformed shapes of the rhombus under tension and for an apex angle of 15° with different $n$-values: (a) $n = 1.0$, (b) $n = 0.7$, (c) $n = 0.5$ and (d) $n = 0.3$.

Fig. 15. Deformed shapes of the rhombus under tension and for an apex angle of 45° with different $n$-values: (a) $n = 1.0$, (b) $n = 0.7$, (c) $n = 0.5$ and (d) $n = 0.3$. 
the rate of increase of the nonlinear stiffness with respect to the load increases as the material constant \( n \) decreases. For a given material constant and for different apex angles, this rate of increase is nearly constant for a good portion of the tensile load as could be observed in Fig. 10. In the compression case, the magnitude of the compressive load is quite limited as compared to the tensile load in the tension case.

The deformed shape of the rhombus frame is plotted for three values of the apex angle, 75°, 45° and 15° and different loading parameters. Figs. 11–13 show the deformed shapes for the compression case and for four different material constants, \( n = 1.0, 0.7, 0.5 \) and 0.3. Similarly, Figs. 14–16 show the deformed shapes for the tension load case. These figures demonstrate the ability of the semi-analytical solution to predict extreme cases of large deflection for the rhombus frame. In addition, these figures indicate that the frame exhibits a spring behavior with useful applications in fields such as compliant mechanisms, flexural pivots, MEMS and others.

7. Conclusions

A semi-analytical solution has been derived for the spring behavior of a prismatic flexible rhombus frame made of a material having a nonlinear stress–strain relationship of the Ludwick type. Different apex angles with four different material constants are considered along with a series of tension and compression loads. The spring behavior of the frame is presented by its load–deflection curve and nonlinear secant stiffness. It is observed from the load–deflection curves that the frame has a hardening effect in tension and a softening effect in compression for all apex angles. The nonlinear secant stiffness has a nearly constant rate of increase for a considerable region in the tension load zone. When compared with finite-element analysis, the semi-analytical solution gave excellent agreement and predicted extreme cases of large deflection.

References