A Model-Based Formulation of Robust Design

1 Introduction

Robust design has been around for about half of a century by now. In fact, Genichi Taguchi, the recognized father of the concept, developed his approach in the postwar years, in which design engineering was based on experience and extremely costly experiments performed on a black box. The engineer could apply inputs to the black box under various laboratory-emulated operation conditions and measure the performance of the system. The engineer could then apply sign engineering was based on experience and extremely costly experiments performed on a black box. The engineer could apply inputs to the black box under various laboratory-emulated operation conditions and measure the performance of the system. The engineer could then apply

Other researchers, in turn, have engaged in extending Taguchi’s method to allow for constrained engineering design problems, which argues that cannot be handled with the current Taguchi methods [5]. With the aid of nonlinear programming, robust design has also been formulated as worst-case scenarios, in which both the robustness of design objectives and constraints are considered [6]. Another research mainstream is oriented toward the response surface methodology and the compromise decision support problem [7].

Along these lines and in connection with mechanical systems, Gadallah and El-Maraghy [8] identified the design sensitivity matrix as the Hessian of the objective function. However, associating the Hessian with the sensitivity matrix is limiting, in that this ties robust design to the objective-function approach, which need not be the case. Moreover, examples of robust design for engineering elements and systems have been reported; applications include internal combustion engines [9], passenger aircraft [10], absorption chillers [11], and mechanisms [12,13]. More recently, Hu et al. [14] reported on the application of robustness to the design of compliant assembly systems using a minimization of the spectral norm of the covariance matrix of the elastic displacements of sheet-metal components. This approach is in line with the formulation proposed in this paper.

Some researchers have employed deterministic methods for approaching robust design with tolerances that are given beforehand [15,16]. However, within these approaches, in order to account for the inherently random nature of the parameters, upper and lower limits on the parameters are introduced, which are supposed to be given. Other researchers have taken into consideration the stochastic nature of the relevant parameters, upper solving a robust design problem assuming that the statistical properties of the parameters are known a priori [17–19]. Consequently, a solution obtained is subject to the statistical properties assumed. However, knowing these properties is not to be taken for granted in most cases.

In this paper, we introduce a formulation of robust design based on the mathematical model available for the object under design, while taking into account the stochastic nature of the parameters. A methodology is then introduced that, when applicable, obviates the a priori knowledge of the statistical properties of the distributions at hand.

It is worth mentioning that the framework proposed here has

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Contributed by the Design Automation Committee for publication in the JOURNAL OF MECHANICAL DESIGN. Manuscript received January 4, 2004; revised June 3, 2004. Associate Editor: W. Chen.
been applied to the shape optimization of machine elements using finite elements and a custom-tailored mesh-generation algorithm [20].

2 Global Versus Local Robust Design

In the context of robust design, we distinguish here between what we shall call global robust design and local robust design. In order to better understand the difference between global and local robust design, we will look first at the former.

In a nutshell, what we need to robustly solve a design problem is a model, which can be of many different kinds. The simplest models to handle are those described by simple functional relations. Most practical decision-making problems, of which design tasks are a subset, are not amenable to such a simple formulation. For example, the decision-making process underlying the creation of the International Alphabet is a paradigm of global robust design. The International Alphabet, adopted among others by the International Civil Aviation Organization (ICAO), was created to allow the transmission of a spoken message with minimum distortion, even in the presence of a variety of accents, dictions, and educational backgrounds. In transmitting an oral message via a noisy communication channel, which involves the vocal system of the speaker, an “A” as such can be mistaken with an “eight.” The signal is strengthened by replacing “A” with “Alpha,” such a mistake thus becoming much less likely.

At the other end of the spectrum we have locally robust designs, which are modelled in the form of equality relations between performance functions—where “function” is used in the mathematical sense—of both the design variables, to which the designer is to assign numerical values, and the design-environment parameters, over which the designer has no control. This paper focuses on locally robust design, which is amenable to a representation by means of analytic functions, or at least smooth functions, of the variables in the design task, endowed with continuous first derivatives, as would be needed for linearization.

3 Formulation of the Robust Design Problem

Engineering design is increasingly becoming model based, in that its complexity calls for a mathematical model involving multiple quantities, some of which are to be decided on by the designer with the purpose of meeting performance specifications—e.g., the thrust that an aircraft engine must deliver at a given rpm—under given environment conditions—engine must operate at a specified ambient temperature and at a given ambient pressure. The aim of robust design is products whose performance remains within specifications in the presence of large variations in environment conditions.

3.1 Classification of Design Quantities. In setting up a framework for robust design, we start by classifying the quantities at play in the design task.

- Design variables (DV) are those quantities to be decided on by the designer with the purpose of meeting performance specifications under given conditions.

We assume that the design task involves \( n \) such variables, which are thus grouped in the design-variable vector \( \mathbf{x} \), i.e.,

\[
\mathbf{x} = [x_1 \ x_2 \ \cdots \ x_n]^T \quad (1)
\]

- Design-environment parameters (DEP) are those quantities over which the designer has no control, and that define the conditions of the environment under which the designed object will operate. These are, for example, ambient temperature and pressure, humidity level, a specific robot task, and so on. These variables, moreover, are of a random nature. We assume that, for the design task at hand, \( v \) DEP describe the environment conditions, and group all these into the design-environment vector \( \mathbf{p} \), namely,

\[
\mathbf{p} = [p_1 \ p_2 \ \cdots \ p_v]^T \quad (2)
\]

The concept of design environment is a bit elusive to grasp, and can be misleading, for which reason further explanation is warranted. The design environment can be quite different from the physical environment. For example, the robot designer has absolutely no control over the specific tasks that his or her design will be commanded to execute. It is thus apparent that the task is the environment for the robot-design job.

Moreover, the concept of design environment is relative: When designing a high-precision cam, for example, with extremely tight tolerances, the cam designer has no control over which machine tool the cam manufacturer will use. In this case, the machine tool is the environment of the cam-design job. By the same token, the machine-tool designer has no control over the workpieces that his or her design will be tasked to execute. In this case, the cam becomes the environment of the machine-tool design job.

- Performance functions (PF) are quantities used to represent the performance of the design in terms of design variables and design-environment parameters. The performance functions will be grouped in the \( m \)-dimensional vector \( \mathbf{f} \), namely,

\[
f = [f_1 \ f_2 \ \cdots \ f_m]^T \quad (3)
\]

Let us assume that the components \( \{x_i\}_1^n \) and \( \{p_i\}_1^v \) of \( \mathbf{x} \) and \( \mathbf{p} \), respectively, have been normalized so that they are all dimensionless, the normalization being done by dividing all physical design variables by their nominal values, in Taguchi’s terminology. We have, in this context, a set of functions \( \{f_i(x;p)\}_1^m \) of the design variables and the design-environment parameters that represent the performance of the object under design. Thus, we have the functional relation

\[
f = f(x;p) \quad (4)
\]

that encapsulates the mathematical model at hand.

Notice that \( m, \ n, \ v \) need not be related in any way. However, if it happens that \( m = n \), then Eq. (4) constitutes a determined system of equations in \( \mathbf{x} \), which means that the design task at hand is too constrained to allow for robust design.

In the formulation below, we assume that the DEP have the same impact on the design, and all PF bear the same importance, i.e., they have identical weights. However, in some cases, trade-offs between DEP and PF must be considered. These can be handled by associating a corresponding weight \( w_i \), \( i = 1, \ldots, v \), to each \( p_i \) of Eq. (2) and its counterpart \( \psi_i, i = 1, \ldots, m \), to each \( f_i \) of Eq. (3). For the sake of conciseness, we refrain from elaborating on this issue in this paper.

3.2 Random Nature of the DEP. Given the randomness of the DEP, we also need models for their probability distributions. In this vein, and for the sake of conciseness, we assume that the variations of the DEP obey Gaussian distributions with nonzero mean and nonidentical standard deviations. One advantage of the Gaussian distribution lies in its simplicity, for it is fully described by two parameters, the mean and the standard deviation. Of course, simplicity per se is of little use to real-world problems. It turns out, however, that the Gaussian distribution faithfully reflects the actual distribution found in practice. Indeed, the justification for representing many complicated phenomena by Gaussian density functions lies in the central limit theorem [21]. If \( x \) is the sum of \( N \) independent random quantities having nonidentical density functions, then \( x \) tends to have a Gaussian density function as \( N \) approaches infinity.

Further, if \( p_0 \) indicates the nominal operating conditions, then the expected value of the variation in DEP, denoted by \( \mu_p \), can be represented as

\[
\mu_p = E[\Delta p] = E[p - p_0] \quad (5)
\]

where \( E[\cdot] \) is the expected-value operator. Moreover, the covariance matrix of \( \Delta p \), denoted by \( \mathbf{P} \), can be evaluated as
\[
P = V[\Delta p] = E[\Delta p - \mu_p](\Delta p - \mu_p)^T = E[\Delta p \Delta p^T] - \mu_p \mu_p^T
\]
(6)

where \(V[\cdot]\) is the covariance operator. Robust design aims at rendering the performance vector \(f\) of a design insensitive to variations \(\Delta p\) as possible. In this vein, we assume that the functional relation of Eq. (4) is differentiable with respect to the DEP, and hence, we have, in light of Eq. (4), and upon expansion around the nominal point \((x; p_0)\)

\[
f(x; p_0 + \Delta p) = f(x; p_0) + \left(\frac{\partial f}{\partial p}\right)_{p = p_0} \Delta p + \text{HOT}
\]
(7)

where \(\text{HOT}\) represents higher-order terms. If we introduce further

\[
\Delta f = f(x; p_0 + \Delta p) - f_0, \quad f_0 = f(x; p_0)
\]
(8)

then, Eq. (7) becomes, upon neglecting \(\text{HOT}\)

\[
\Delta f = F \Delta p, \quad F = \left(\frac{\partial f}{\partial p}\right)_{p = p_0} \in \mathbb{R}^{m \times p}
\]
(9)

Obviously, \(f_0\) is the nominal performance function vector evaluated at the nominal value \(p_0\). Moreover, \(F\) is the \(m \times p\) Jacobian matrix of \(f\) with respect to \(p\). Thus, \(F\) measures the sensitivity of the design performance to variations in the design-environment parameters, and will be called the performance matrix of the design at hand.

Now, the expected value of \(\Delta f\) is

\[
\mu_f = E[\Delta f] = F \mu_p
\]
(10)

which follows by virtue of the linear relation, Eq. (9). Moreover, the corresponding covariance matrix \(\Phi\) of \(\Delta f\) is evaluated as

\[
\Phi = V[\Delta f] = E[(\Delta f - \mu_f)(\Delta f - \mu_f)^T] = E[\Delta f \Delta f^T] - \mu_f \mu_f^T
\]
(11)

Substituting Eqs. (9) and (10) into Eq. (11), we obtain

\[
\Phi = E(F(\Delta p - \mu_p)(\Delta p - \mu_p)^T F^T) \equiv F E(\Delta p \Delta p^T) F^T
\]
(12)

which can be simplified to

\[
\Phi = F E(\Delta p \Delta p^T) F^T
\]
(13)

Recalling the definition of \(P\) of Eq. (6), the above expression of \(\Phi\) simplifies to

\[
\Phi = FF^T
\]
(14)

We will dwell below on three cases, depending on the dimensions of vectors \(f\) and \(p\).

3.2.1 Single PF and Single DEP. Under a single performance function and a single DEP, the performance matrix turns out to be a scalar, Eq. (9) thus simplifying to the form

\[
\Delta f = F \Delta p, \quad F = \left(\frac{\partial f}{\partial p}\right)_{p = p_0} \in \mathbb{R}
\]
(15)

The mean and variance of \(\Delta f\) are, in turn

\[
\mu_f = F \mu_p, \quad \sigma_f = F^2 E[\Delta p^T] - \mu_p^2
\]
(16)

where \(\mu_f\) and \(F\) are now scalars. Moreover, the relation

\[
\sigma_p = E[\Delta p^2] - \mu_p^2
\]
(17)

follows as a special case of Eq. (6), and \(\sigma_f\) can be expressed as

\[
\sigma_f = F^2 \sigma_p
\]
(18)

Robust design thus reduces to finding the minimum of \(\sigma_f\), which is the product of two factors, \(F^2\) and \(\sigma_p\). Obviously, the designer has no control over \(\sigma_p\), and hence, the robust design problem reduces to the minimization of \(F^2\), as defined in Eq. (15); that is

\[
z(x) = F^2 \rightarrow \min_x
\]
(19a)

subject to

\[
f(x; p_0) = f_0
\]
(19b)

Accordingly, if the PF and DEP are both single, robust design can be strictly secured by directly minimizing the variance of the performance function without a priori knowledge of the statistical properties of the DEP.

3.2.2 Single PF and Multi-DEP. In the case of a single performance function and multi-DEP, the performance matrix reduces to a \(1 \times p\) matrix \(F\), i.e.,

\[
\Delta f = F \Delta p, \quad F = \left(\frac{\partial f}{\partial p}\right)_{p = p_0} \in \mathbb{R}^{1 \times p}
\]
(20)

In this case, the variance of \(\Delta f\) is a scalar, as in the previous case, namely

\[
\sigma_f = E(F(\Delta p \Delta p^T - \mu_p \mu_p^T) F^T) = F E[\Delta p \Delta p^T] - \mu_p \mu_p^T F^T
\]
(21)

Using Eq. (6), the above expression of \(\sigma_f\) reduces to

\[
\sigma_f = F F^T \|\|_F
\]
(22)

which \(F\) denotes the single row of \(F\) in vector form, i.e., as a column array, and \(\|\|_F\) is the weighted Euclidean norm of vector (\(\cdot\)) with respect to the positive-definite matrix \(P\). Moreover, we can write

\[
\|\|_F = \|F\|_F^2 = \text{tr}(F^T F) = \text{tr}(F F^T) = \text{tr}(P) - \|\|_F^2
\]
(23)

in which \(\|\|_2\) is the Euclidean norm of (\(\cdot\)), while tr(\(\cdot\)) indicates the trace of matrix (\(\cdot\)). Notice that, in deriving the above relation, we recalled a trace inequality of positive-definite matrices [22]. Since the designer has no control over \(P\), minimizing the weighted norm \(\|\|_F\) reduces to minimizing \(\|\|_F^2 = \|F\|_F^2 = \|F\|_F = \|F\|_F^2\). That is, the robust design problem can be formulated as

\[
z(x) = \frac{1}{2} F F^T \rightarrow \min_x
\]
(24a)

subject to

\[
f(x; p_0) = f_0
\]
(24b)

for given \(p_0\) and \(f_0\).

3.2.3 Multi-PF and Multi-DEP. Now, we address the general case in which a multiperformance function and a multi-DEP are present. Here, we consider any norm of the covariance of \(\Delta f\) and denote it by \(\sigma_f\), that is

\[
\sigma_f = E[(\Delta f - \mu_f)(\Delta f - \mu_f)^T] = \|\Phi\|_F
\]
(25)

where \(\|\|_F\) indicates any norm of its argument (\(\cdot\)). Notice that Eq. (11) was used to obtain the above expression. Thus, using the definition of \(\Phi\) of Eq. (12), \(\sigma_f\) of Eq. (25) can be expressed as

\[
\sigma_f = \|FF^T\|_F
\]
(26)

Therefore, for the purpose of achieving a robust design, we aim at minimizing \(\sigma_f\), i.e., we want \(\|FF^T\|_F\) to be a minimum. Formally, this can be stated as a constrained matrix-norm minimization problem of the form

\[
z(x) = \|FF^T\|_F \rightarrow \min_x
\]
(27a)

subject to

\[
f(x; p_0) = f_0
\]
(27b)

Hence, the general locally robust design problem can be formulated as one of the minimization of a norm of the covariance matrix of the variations in the PF upon variations in the DEP, as formally stated in Eqs. (27a) and (27b).
Now, to solve the optimization problem, Eq. (27), we need first to adopt a matrix norm, which should be consistent with the Euclidean vector norm adopted here. The two consistent norms are the 2-norm, denoted by $\| \cdot \|_2$, and the Frobenius norm, denoted by $\| \cdot \|_F$. Their basic definitions and properties can be found in [23].

In fact, the above-mentioned matrix norms can be expressed in terms of the matrix singular values, namely, for any matrix $A$

$$\| A \|_2 = \max_i (s_i) \quad \| A \|_F = (\sum_i s_i^2)^{1/2}$$

(28)

in which $\{s_i\}$ is the set of nonzero singular values of $A$ and $r = \text{rank}(A)$. Thus, the 2-norm of a matrix, also known as the spectral norm, is its maximum singular value. Unfortunately, this norm is expensive to compute, besides being, in general, a nonanalytic function of the matrix at hand: If $A$ changes under changes of a parameter $p$, and its singular values are ordered as $s_1 \leq s_2 \leq \cdots \leq s_r$, at a given value of $p$, then at a slightly different value of $p$ this ordering is not preserved. That is, the maximum singular value can jump from index to index—singular-value index.

Therefore, if the 2-norm is used to evaluate the matrix norm of Eq. (27), then the objective function of this optimization problem, in general, is nondifferentiable. Thus, the minimization procedure used to solve Eqs. (27) is limited to direct-search methods, i.e., minimization methods based on evaluations of the objective function only.

On the other hand, the most important matrix norm that is not induced by a vector norm is the Hilbert-Schmidt or Frobenius norm. This norm, introduced in Eq. (28), can be defined more compactly and more generally as

$$\| A \|_F = \sqrt{\text{tr}(A^TWA)} \quad W = \frac{1}{n} I_{n \times n}$$

(29)

where $I_{n \times n}$ is the $n \times n$ identity matrix. A nice feature of the matrix Frobenius norm is an analytic function of the matrix at hand, which makes it an attractive alternative to the 2-norm. In fact, the matrix Frobenius norm is an analytic function of the matrix at hand: If $A$ changes under changes of a parameter $p$, and its singular values are ordered as $s_1 \leq s_2 \leq \cdots \leq s_r$, at a given value of $p$, then at a slightly different value of $p$ this ordering is still preserved. That is, the singular values can jump from index to index—singular-value index.

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where $I_{n \times n}$ is the $n \times n$ identity matrix. A nice feature of the matrix Frobenius norm is an analytic function of the matrix at hand, which makes it an attractive alternative to the 2-norm. Indeed, the Frobenius norm allows a straightforward derivation of expressions of gradients of the objective function with respect to the design variables.

If the Frobenius norm is adopted, then the expression for $\sigma_f$ of Eq. (26) reduces to

$$\sigma_f = \| FP \|_F = \left[ \frac{1}{n} \text{tr}(FP^T) \right]^{1/2}$$

(30)

which can be simplified further to

$$\sigma_f = \left[ \frac{1}{n} \text{tr}(F^T F) \right]^{1/2} = \left[ \frac{1}{n} \text{tr}(F^T F) \right]^{1/2}$$

(31)

where a property of the trace of a product of matrices has been recalled: $\text{tr}(ABC) = \text{tr}(CAB) = \text{tr}(BCA)$, for matrices $A$, $B$, and $C$ compatible under multiplication. Notice that minimizing $\sigma_f$ as given above requires knowing $P$, which is not always available. In the absence of knowledge of $P$, we can instead minimize an upper bound of $\sigma_f$, obtained from Eq. (31) upon inverting the trace inequality that we recalled when we derived Eq. (23), namely

$$\sigma_f \leq \sqrt{\frac{1}{n} \text{tr}(F^T F)} = \sqrt{\frac{1}{n} \text{tr}(F^T F)} = \sqrt{\frac{1}{n} \text{tr}(F^T F) \text{tr}(P)}$$

(32)

Consequently, the general locally robust design problem can be formulated as one of the minimization of the bound given by Eq. (32), which comprises two factors, $\text{tr}(F^T F)$ and $\text{tr}(P)$. The designer not having any control over the second factor, the minimization problem at hand reduces to

$$z(x) = \text{tr}(F^T F) \rightarrow \min_x$$

(33a)

subject to

$$f(x;p_0) = f_0$$

(33b)

Nevertheless, it is still possible to minimize $\sigma_f$ itself, rather than a bound thereof—a bound leads to a conservative-solution—under certain conditions, as discussed below.

4 The Case of Isotropic Design

Under special conditions, which are not difficult to find in certain application domains like robotics, the robust design problem admits a tighter formulation, not depending on any upper bound of $\sigma_f$.

In this case, we will follow an alternative procedure to obtain a robust design: First, render the performance matrix isotropic—all its singular values identical—and then, minimize its resulting common singular value. Here, we limit the discussion to cases in which it is possible to meet conditions that render $F$ isotropic.

Imposing the isotropy conditions is advantageous. Indeed, what isotropy brings about is a design whose performance is homogeneous over the DEP space; that is, any DEP disturbance of a given norm will have identical impact on the PF, regardless of the direction of said disturbance. More precisely, under an isotropic performance matrix, a disturbance $\Delta p$ in the DEP of a given Euclidean norm $\| \Delta p \|_2$, produces a variation $\Delta f$ in the performance function of a constant Euclidean norm $\| \Delta f \|_2$, regardless of the direction of $\Delta p$.

Consequently, the design performance variation obtained by the methodology proposed here will be both a minimum and independent of a particular $\Delta p$ at the same time. In other words, the sensitivity, in the presence of isotropy, remains a minimum over the whole domain of $\Delta p$, for a given value of $\| \Delta p \|_2$. In fact, when an isotropic $F$ is at all possible, the general robust design problem becomes as simple as the single-PF-single-DEP case.

Let $S = F^T F$ be the $v \times v$ sensitivity matrix, which, from Eq. (31), apparently plays an important role in locally robust design. Notice that $S$ is symmetric and positive-semidefinite, which brings us to the discussion of rank($S$).

First and foremost, notice that, except for special values of its arguments, which can render $F$ rank-deficient, this matrix is, in general, of full-rank, i.e.,

$$\text{rank}(F) = r = \min\{m, v\}$$

(34)

Indeed, $F$ can be identically rank-deficient, i.e., for any values of its arguments, only under two possibilities: (i) some DEP have no effect on the performance vector $f$, and hence, can be deleted from the components of $p$; and (ii) some of the performance functions are not affected by the DEP, and hence, can be deleted from the components of $f$. That is, $F$ can be identically rank-deficient only if superfluous variables are used. If this is not the case, which we will assume henceforth, then rank($F$) is given by Eq. (34), except for isolated values of its arguments, as we will illustrate with one example.

Now, if $f$ is isotropic, then its $r$ nonzero singular values $\{\sigma_i\}_1^r$ are identical, i.e.,

$$\sigma = \sigma_1 = \cdots = \sigma_r$$

(35)

We distinguish two cases, depending on the rank of $F$:

Case I: $r = v \leq m$. In this case, the sensitivity matrix $S$ turns out to be a multiple of the $v \times v$ identity matrix, $I_{v \times v}$, i.e.,

$$S = F^T F = \sigma^2 I_{v \times v}$$

(36)

and hence, from Eq. (31), the expression for $\sigma_f$ becomes, under the isotropy condition, Eq. (35)

$$\sigma_f = \sigma \sqrt{\frac{1}{n} \text{tr}(P)}$$

(37)

which is the product of two factors—the first one, $\sigma$, being a function of $x$, the DV vector; the second is independent of $x$ and
lies beyond the control of the designer. Hence, if matrix $F$ can be rendered isotropic, when $\nu \leq m$, the robust design problem reduces to

$$\sigma = \sigma(x) \rightarrow \min_x$$

subject to

$$F^TF = \sigma^2 I_{\nu \times \nu} \quad \text{and} \quad f(x; p_0) = f_0 \quad (38b)$$

Case 2: $r = m \leq \nu$. In this case, matrix $S$, by virtue of Eq. (35), can be reduced to

$$S = F^T F = \sigma^2 \begin{bmatrix} I_{m \times m} & O_{m \times \nu'} \\ O_{\nu' \times m} & O_{\nu' \times \nu'} \end{bmatrix} \quad (39)$$

where $\nu' = \nu - m$ and $O_{k \times l}$ denotes the $k \times l$ zero matrix. Hence, $\sigma_f$ of Eq. (31) becomes

$$\sigma_f = \sqrt{\frac{1}{m} \sum_{i=1}^{m} q_{ii}} \quad (40)$$

where $q_{ii}$ denotes the $i$th diagonal entry of $P^2$. Again, $\sigma_f$ turns out to be the product of two factors, only the first one of which is a function of $x$. Hence, the robust-design problem reduces to the minimization problem of Eq. (38a), subject to Eqs. (27b) and (39).

### 4.1 A Geometric Interpretation of Design Sensitivity

The scalar variance in a direction indicated by a deterministic unit vector $u$ is given by

$$V(u^T \Delta f) = E[(u^T \Delta f)(u^T \Delta f)^T]$$

Upon simplifying, we obtain

$$V(u^T \Delta f) = u^T E[\Delta f \Delta f^T] u$$

Using Eq. (26), the above equation can be rewritten as

$$V(u^T \Delta f) = u^T FPF^T u = u^T \Phi u \quad \Phi = FPF^T \quad (43)$$

If we let $u = \Delta p$, which is possible because $p$ has been assumed dimensionless, then we can define the scalar variance of the design in that direction as

$$S^2 = V[\Delta p^T \Delta f] = \Delta p^T \Phi \Delta p \quad (44)$$

The covariance matrix $\Phi$ can be written in diagonal form as

$$\Phi = FPF^T = E^T \Lambda E \quad \text{or} \quad E^T \Lambda E^T = \Lambda$$

where the columns of matrix $E$ are the $\nu$ mutually-orthogonal eigenvectors of $\Phi$ and $\Lambda$ is a $\nu \times \nu$ diagonal matrix whose diagonal entries are the real, positive eigenvalues of the same matrix $\Phi$, i.e.,

$$E = [e_1, e_2, \cdots, e_\nu], \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_\nu) \quad (46)$$

Moreover, if we let

$$\Delta p = E^T y \quad (47)$$

then, Eq. (44) can be cast in the form

$$S^2 = y^T E \Phi E^T y = y^T \Lambda y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_\nu y_\nu^2 \quad (48)$$

Geometrically, Eq. (48) has an interesting interpretation: Let us fix $S$ to a value, say $S_0$, Eq. (48) then leading to

$$\frac{y_1^2}{(S_0/\lambda_1)^2} + \frac{y_2^2}{(S_0/\lambda_2)^2} + \cdots + \frac{y_\nu^2}{(S_0/\lambda_\nu)^2} = 1 \quad (49)$$

This equation represents a $\nu$-dimensional ellipsoid with center at the origin of the $\Delta p$-space. This ellipsoid is called here the design-sensitivity ellipsoid, the direction of its principal axis $y_i$ being that of the corresponding eigenvector $e_i$. Moreover, the length of the $i$th principal axis is $S_0/\lambda_i$. Shown in Fig. 1 are two instances of a three-dimensional design-sensitivity ellipsoid.

Because different points on the ellipsoid surface lead to the same design-performance variance, the larger the value of $\|\Delta p\|_2$ is along a direction, the less sensitive the performance to the variation of the DEP will be in that direction. Since the largest and the smallest values of $\|\Delta p\|_2$ occur at the longest and shortest axes of the ellipsoid, respectively, the performance is least sensitive to the variation along the direction of the largest semiaxis and most sensitive along the direction of the shortest semiaxis, as depicted in Fig. 1a for a three-dimensional sensitivity ellipsoid.

Shown in Fig. 1b is an “ellipsoid” of identical axis lengths, i.e., a sphere. This particular case represents an isotropic matrix $\Phi$, which is different from the design-sensitivity matrix. Notice that achieving an isotropic $\Phi$ requires knowledge of $P$. Since $P$ is not always available, we focus below on the isotropy of $F$.

Any set of DV leading to an isotropic design performance matrix $F$ will be said here to produce an isotropic design.

From the above discussion, an indicator of how far a design lies from isotropy can be assessed by the ratio of the largest $\sigma_f$ to the smallest $\sigma_f$ singular values of $F$, i.e., $\sigma_f/\sigma_{f,\min}$, This ratio happens to be one form of defining the condition number of the performance matrix $F$. Indeed, the condition number of this matrix, based on the matrix 2-norm [23], is defined as

$$\kappa_2(F) = \frac{\sigma_f}{\sigma_{f,\min}} \quad (50)$$

The above definition of the matrix condition number poses challenging problems to the designer when, for example, trying to minimize it over certain parameters. Indeed, the largest or, for that matter, the smallest singular values of $F$ are, respectively, the square roots of the largest and the smallest eigenvalues of $S$, which is a square matrix. Now, the largest and the smallest eigenvalues of a square matrix are not analytic functions of the matrix entries [23]. This means that explicit formulas for their derivatives with respect to any scalar parameter are elusive. The good news is that the matrix condition number lends itself to alternative definitions, the one best suiting our aims being the general definition, if based on the matrix Frobenius norm. The general definition of the condition number of $F$ is recalled below

$$\kappa(F) = \frac{\|F\|\|F^{-1}\|}{\|F\|\|F^{-1}\|} \quad (51)$$

where $\kappa(\cdot)$ is the condition number based on any norm $\|\cdot\|$ of the matrix argument $\cdot$, while $F^{-1}$ is the generalized inverse of $F$. Indeed, if the Frobenius norm is used in the definition of the condition number, then we have

$$\kappa(F) = \frac{1}{n} \sqrt{\text{tr}(FF^T) \text{tr}(FF^T)^{-1}} \quad (52)$$
5 Examples

In order to strengthen the concepts and illustrate the application of the methodology proposed here, we include below some engineering examples in robust design.

Example 1: Robust Design of a Belt

Belts are used in the transmission of power between shafts with either parallel or skewed axes. The power transmitted by a belt is given by [24]

\[ P = (1 - e^{-\mu \theta})(T - MV^2)V \]  

where

- \( M \) = the mass of the belt per unit length
- \( V \) = the belt speed
- \( \mu \) = the coefficient of friction
- \( \theta \) = the contact angle
- \( T \) = the tension in the belt
- \( P \) = the transmitted power

The design variable, design-environment parameter, and the performance function are

\[ x = M, \quad p = V, \quad f = P \]

In this case, both the performance function \( P \) and the design environment parameter \( V \) are scalars, which implies a design job falling in the case discussed in Subsection 3.2.1. Thus, minimizing the variance \( \sigma_P \) of \( P \) reduces to minimizing the squared first derivative of the power capacity of the belt \( P \) with respect to \( V \).

Now, to find the variation in the power \( P \) due to a variation in the operating belt speed \( V \), we use Eq. (53) to write

\[ \frac{\Delta P}{P} = F \frac{\Delta V}{V}, \quad F = \frac{a - 3}{a - 1}, \quad a = \frac{T}{MV^2} \]

then, we aim at solving

\[ F^2 = \left( \frac{a - 3}{a - 1} \right)^2 \rightarrow \min_a \]

which thus leads to

\[ a = 3 \Rightarrow T = 3MV^2 \]

and hence, the optimum value of \( M \) is \( M = T/(3V^2) \).

Example 2: Robust Design of a Pneumatic Cylinder

Consider a pneumatic cylinder used to move a load of weight \( W \) along a horizontal surface, as shown in Fig. 2, the friction force between the load and the surface being \( F \). The acceleration of the load is to occur within a distance \( L \) and the load is to attain a steady-state velocity \( V \) thereafter. If the supply pressure is \( P \), determine the actuator size \( D \) required, the distance \( L \), and the pressure \( P \) for a robust design.

The design variable vector, the design-environment vector, and the performance function are

\[ x = \begin{bmatrix} L \\ D \\ P \end{bmatrix}, \quad p = \begin{bmatrix} W \\ F \end{bmatrix}, \quad f = V \]

where, obviously, the performance function is one-dimensional, i.e., a scalar. It is required that the steady-state velocity of the cylinder over a stroke \( L \) satisfy the relation [25]

\[ V = \sqrt{\frac{gL(\pi D^2P - 4F)}{2W}} \geq 0, \quad \text{under} \quad \frac{\pi D^2P}{4F} \geq 1 \]

which yields the performance function

\[ f = \sqrt{\frac{gL(\pi D^2P - 4F)}{2W}} \]  

(54)

Obviously, this case is of the kind studied in Subsection 3.2.2. Accordingly, to minimize the variance \( \sigma_f \), we must minimize the norm of the gradient vector of \( V \) with respect to both \( W \) and \( F \).

From the performance function, Eq. (54), we can readily write

\[ \frac{\Delta V}{V} = F \left[ \frac{\Delta W/W}{\Delta F/F} \right], \quad F = \left[ -\frac{1}{2} - 2F/(\pi D^2P - 4F) \right] \]

Obviously, the performance matrix \( F \) in this case turns out to be of \( 1 \times 2 \), with its norm \( \sigma \) being given by

\[ \sigma^2 = F^T \frac{1}{4} \left( \frac{1}{(\lambda - 2)^2} \right)^2, \quad \lambda = \frac{\pi D^2P}{2F} \geq 2 \]  

(55)

Thus, the robust design of the pneumatic actuator, in the absence of knowledge of the statistical properties of \( \Delta W \) and \( \Delta F \), can be formally stated as a minimization problem, namely

\[ \sigma^2 = \frac{1}{4} \left( \frac{1}{(\lambda - 2)^2} \right)^2 \rightarrow \min \lambda \]  

(56a)

subject to

\[ V_0 = \frac{gL}{4W_0F_0} (\lambda - 2), \quad 2 \leq \lambda \leq \lambda_{max} \]  

(56b)

for given nominal values \( V_0 \), \( W_0 \), and \( F_0 \).

By virtue of the constraints Eq. (56b) and in light of Eq. (55), the objective function takes its minimum at \( \lambda = \lambda_o \) defined as

\[ \lambda_o = \frac{\pi D^2_{min}P_{min}}{2F_o} \]

where \( D_{min} \) and \( P_{min} \) denote the minimum possible values of \( D \) and \( P \), respectively, the corresponding value of \( L \) then being obtained from Eq. (56b) as

\[ L_o = \frac{4V_0^2W_0F_0}{\lambda_o - 2} \]

Example 3: Robust Design of a Bar Under Torsion and Tension

Now, we consider a case in which both the PF and DEP are multiple, still, in the absence of statistical data on the variations of the DEP. Let us consider a circular bar of radius \( d \) and length \( L \) under an axial force \( P \) and a torque \( T \), as shown in Fig. 3. The performance of this bar is measured by the twist angle \( \theta \) and the axial displacement \( u \). It is required to find \( D \) that will make the design robust under variations of \( P \) and \( T \). The performance function and design environment parameter vectors in this case are

\[ f = \begin{bmatrix} \theta \\ u \end{bmatrix}, \quad p = \begin{bmatrix} T \\ P \end{bmatrix}, \quad x = \begin{bmatrix} L \\ D \end{bmatrix} \]

The relevant relations are, moreover

\[ \theta = \frac{32LT}{\pi GD^3}, \quad u = \frac{4LP}{\pi ED^4} \]  

(57)

where \( G \) is the modulus of rigidity of the bar.

From Eqs. (57), we obtain
and hence,
\[ \begin{bmatrix} \Delta \theta \\ \Delta u / u_0 \end{bmatrix} = F \begin{bmatrix} \Delta T / T \\ \Delta P / P \end{bmatrix} \]

with
\[ F = \sigma^2 \text{diag} \left( \frac{8T}{GD}, \frac{P_0}{E u_0} \right), \quad \sigma^2 = \frac{4L}{\pi D^2} \]

Here we attempt an isotropic design, which requires that the diagonal elements of \( F \) be identical, i.e.,
\[ \frac{8T_0}{GD} = \frac{P_0}{E u_0} \]

Solving for \( D \), we obtain the optimum value \( D_o \) as
\[ D_o = \sqrt{\frac{8T_0 u_0}{\pi G P_0}} \]

and hence, \( F \) turns out to be
\[ F = \sigma^2 I_{2 \times 2}, \quad \sigma^2 = \frac{G}{2 \pi T_0} \left( \frac{P_0}{E u_0} \right)^2 \]

Consequently, for a robust design, all we need is to make \( L \) as small as possible, while respecting the isotropy condition, Eq. (58).

**Example 4: The Robust Design of a Helical Spring**

Figure 4 shows a helical spring loaded by the axial force \( F \). We denote by \( D \) the mean spring diameter, by \( d \) the wire diameter, and by \( N \) the number of turns. In addition, the stiffness \( k \) of the spring, the shear stress \( \tau \) in the spring, and the natural frequency \( \omega_n \) of the spring are given by
\[ k = \frac{d^4 G}{8D^3 N}, \quad \tau = K_i \frac{8FD}{\pi d^2}, \quad \omega_n = \frac{1}{2} \sqrt{\frac{k}{M}} \]

where \( G \) is the shear modulus, \( M \) is the mass of the spring, and \( K_i \) is the shear stress correction factor. Based on the well-known frequency response of a harmonic oscillator, we have
\[ X_o \frac{k}{F_o} = \frac{1}{1 - \gamma^2} \]

where \( X_o \) and \( F_o \) are the displacement and the force amplitudes and \( \gamma = \omega_o / \omega_n \) is the frequency ratio, with \( \omega_o \) denoting the frequency of the harmonic excitation \( F(t) = F_o \cos \omega_o t \). Apparently, the variables over which the designer has no control are \( F_o \) and \( \omega_o \), which are thus the DEP. Thus, the design vector, the DEP vector, and the performance vector are now
\[ x = \begin{bmatrix} D \\ N \end{bmatrix}, \quad p = \begin{bmatrix} F_o \\ \omega_o \end{bmatrix}, \quad f = \begin{bmatrix} \tau \\ X_o \end{bmatrix} \]

Moreover, from Eqs. (59a) and (59b), we can write
\[ \Delta \tau = \Delta F_o \frac{\Delta X_o}{X_o} + \frac{2 \gamma}{1 - \gamma^2} \Delta \omega_o + \frac{2 \gamma^2}{1 - \gamma^2} \Delta \omega_o \]

In this case, the variation in the DEP vector, whose statistical properties are unknown, induces a variation in the performance vector given, to a linear approximation, by
\[ \begin{bmatrix} \Delta \tau / \tau \\ \Delta X_o / X_o \end{bmatrix} = F \begin{bmatrix} \Delta F_o / F_o \\ \Delta \omega_o / \omega_o \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 \\ 1 & 2 \gamma^2 / (1 - \gamma^2) \end{bmatrix} \]

Now, the squared Frobenius norm of \( F \) can be readily obtained as
\[ \| F \|^2_F = 1 + \frac{2 \gamma^4}{(1 - \gamma^2)^2} \]

Figure 5 shows a plot of \( \| F \|^2_F \) versus the frequency ratio \( \gamma \). It can be noted that \( \| F \|^2_F \to \infty \), when \( \gamma = 1 \), at which value \( F \) is singular. This is a case in which the performance matrix is of full rank everywhere, except for the singularity \( \gamma = 1 \), which comes as no surprise, for this case corresponds to resonance. Note also that \( \| F \|^2_F \) remains essentially constant for \( \gamma \approx 8 \), thereby corroborating the established practice: experts design springs for a value of \( \gamma \) of the order of 10.

**Example 5: Robust Design of a Low-Pass Filter**

We consider here the robust design of the RC circuit shown in Fig. 6, first studied in [3] and then in [26]. Both Taguchi and Wilde did not consider the impact of randomness of the design-environment.
parameters. We revisit, here, this problem under two conditions, first without and then with knowledge of the statistics of the variations of the DEP.

The design variables in this case are the resistance $R$ and the inductance $L$, to be determined with the aim of keeping the current $I$ at a nominal value $I_0$ of 10 amperes, while the amplitude $V$ of the excitation voltage $v(t) = V \cos \omega t$ and its frequency $\omega$ undergo variations beyond the control of the designer.

For this filter, the steady-state current $i(t)$ is harmonic of the form $i(t) = I \cos(\omega t + \phi)$, with $I$ and $\phi$ denoting the amplitude and the phase of $i(t)$. These items are given by

$$ I = \frac{V}{\sqrt{R^2 + \omega^2 L^2}}, \quad \phi = \tan^{-1}\left(\frac{\omega L}{R}\right) \tag{62} $$

The DV, DEP, and PF vectors are, respectively,

$$ x = \begin{bmatrix} R \\ L \end{bmatrix}, \quad p = \begin{bmatrix} V \\ \omega \end{bmatrix}, \quad f = \begin{bmatrix} I \\ \phi \end{bmatrix} $$

The corresponding nondimensional variations of vectors $p$ and $f$ are, respectively,

$$ \Delta p = \begin{bmatrix} \Delta I/I_0 \\ \Delta \phi \end{bmatrix}, \quad \Delta f = \begin{bmatrix} \Delta V/V_0 \\ \Delta \omega/\omega_0 \end{bmatrix} $$

In the absence of knowledge of the statistical properties of $\Delta V$ and $\Delta \omega$, the robust-design task is conducted based only on the performance and sensitivity matrices, which are given, respectively, as

$$ F = \begin{bmatrix} 1 & -\alpha^2 \\ 0 & \alpha \sqrt{1-\alpha^2} \end{bmatrix}, \quad S = F^T F = \begin{bmatrix} 1 & -\alpha^2 \\ -\alpha^2 & \alpha^2 \end{bmatrix} \tag{63a} $$

with

$$ \alpha = \frac{I_0 \omega_0}{V} \quad L > 0 \tag{63b} $$

where $V$ is the nominal value of $V$.

The eigenvalues of $S$ are, moreover,

$$ \lambda_{\max} = \frac{1 + \alpha^2 + \sqrt{1 - 2 \alpha^2 + 5 \alpha^4}}{2} > \lambda_{\min} \quad \lambda_{\min} = \frac{1 + \alpha^2 - \sqrt{1 - 2 \alpha^2 + 5 \alpha^4}}{2} > 0 $$

Apparently, an isotropic $F$ requires a vanishing radical in the above expressions. However, the roots of the radical are $\pm \sqrt{0.2 \pm 0.4}$, which are complex, and hence, $F$ cannot be rendered isotropic for any real $\alpha$, which, additionally, should be positive.

Thus, we lack information on $P$ and fail to render $F$ isotropic. What we can do is just minimize the upper bound of $\sigma_f$, as given by Eq. (32), that is,

$$ \text{tr}(F^T F) = 1 + \alpha^2 \tag{64} $$

which takes its minimum at a prescribed lower bound of $\alpha$, denoted by $\alpha_{\min}$.

Now, we consider the case in which the covariance matrix is available from field data, as kindly provided by Dr. M. Huneault, and Dr. D. Asber, of the Institut de Recherche d’Hydro-Québec (IREQ), Canada. The raw data provided to us yield

$$ \bar{V} = 230 \text{ kV}, \quad \sigma_V = 2.8476 \text{ kV}, \quad \bar{\omega} = 60 \text{ Hz}, \quad \sigma_\omega = 0.0304 \text{ Hz} $$

When voltage values are considered at the low end of a transformer assumed to be conservative and delivering power to the consumer at 110 V, we have

$$ V = 110 \text{ V}, \quad \sigma_V = 1.3619 \text{ V}, \quad \bar{\omega} = 60 \text{ Hz}, \quad \sigma_\omega = 0.0304 \text{ Hz} $$

The covariance matrix in this case turns out to be, in dimensionless form,

$$ P = \text{diag}(P_1, P_2) = \text{diag}\left(\frac{\sigma_V^2}{V^2}, \frac{\sigma_\omega^2}{\omega^2}\right) $$

$$ = \text{diag}(1.5329 \times 10^{-4}, 2.5671 \times 10^{-7}) $$

The corresponding covariance matrix $\Phi$ can be evaluated as

$$ \Phi = FPF^T = \begin{bmatrix} P_1 + \alpha^2 P_2 & -\alpha^3 \sqrt{1 - \alpha^2} P_2 \\ -\alpha^3 \sqrt{1 - \alpha^2} P_2 & \alpha^2 (1 - \alpha^2) P_2 \end{bmatrix} \tag{65} $$

Now, the squared Frobenius norm of $\Phi$ can be readily obtained as

$$ \|\Phi\|^2_F = P_1^2 + (2 P_1 + P_2) P_2 \alpha^4 $$

Thus, the robust design of the low-pass filter at hand can be stated as

$$ z(\alpha) = P_1^2 + (2 P_1 + P_2) P_2 \alpha^4 \rightarrow \min_\alpha $$

subject to prescribed bounds

$$ \alpha_{\min} \leq \alpha \leq \alpha_{\max} \tag{66a} $$

Figure 7 depicts the corresponding variation of $\|\Phi\|^2_F$ versus $\alpha$.  

![Fig. 6 A low-pass filter](image_url)

![Fig. 7 Variation of $|\Phi|^2_F$ versus $\alpha$](image_url)
Obviously, $\alpha$ should be chosen as small as possible, i.e., as $\alpha = \alpha_{\min}$, the corresponding optimum values of $L$ and $R$ being readily evaluated as

$$L_0 = \frac{V_0}{I_0 \alpha_0 \alpha_{\min}}, \quad R_0 = \left(\frac{V_0}{I_0} \right)^2 - \frac{\omega_0^2 L_0^2}{\alpha_{\min}^2}$$

It is worth mentioning that the value reported by Taguchi ($\alpha_T = 0.3456$) is smaller than Wilde’s ($\alpha_W = 0.3767$). As a result, if the statistical properties of voltage frequency and amplitude are taken into account, Taguchi’s solution is still more robust than Wilde’s.

6 Conclusions

The robust design problem can be addressed globally or locally. Globally robust design pertains to the preliminary design stage, whereby a detailed mathematical model is not available, but rather a few design alternatives are to be considered. Hence, this stage is not amenable to linearization, which is the basis of locally robust design, the focus of this paper.

A model-based theoretical framework for locally robust design, as applied to engineering systems, is set up through the minimization of the sensitivity of the design performance against environment conditions, while considering the stochastic nature of the design-environment parameters. We showed that minimum sensitivity, and hence, maximum robustness, can be achieved by means of minimizing a norm of the covariance matrix of the design performance functions.

While knowledge of the statistical properties of the variations of the DEP allows for tight robust design solutions, means of dispensing with such knowledge were discussed that are based on the minimization of an upper bound of the design sensitivity.

One special case to guaranteeing robustness in a design is based on isotropy. What isotropy, when at all possible, brings about is a decoupling of the design variables from the covariance matrix of the variations in the DEP, thereby allowing for tight robust design solutions, even if that matrix is not known.

Acknowledgments

The work reported here was supported by NSERC (Canada’s Natural Sciences and Engineering Research Council) under Strategic Project No. 215729-98. The authors wish to express their sincere appreciation to Dr. M. Huneault, and Dr. D. Asher, Institut de Recherche d’Hydro-Quebec (IREQ), Canada, for their help in providing field data characterizing the variations of voltage amplitude and frequency in an undisclosed power system, not necessarily of the Hydro-Quebec network.

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