Central simple superalgebras with anti-automorphisms of order two of the first kind

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ABSTRACT

By a theorem of Albert’s, a central simple associative algebra has an involution of the first kind if and only if it is of order 2 in the Brauer group. Our main purpose is to develop the theory of existence of anti-automorphisms of order 2 of the first kind on finite dimensional central simple associative superalgebras over $K$, where $K$ is a field of arbitrary characteristic. First we need to generalize the Skolem–Noether Theorem to the superalgebra case. Then we show which kind of finite dimensional central simple superalgebras have an anti-automorphism of order 2 of the first kind.

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1. Introduction

An associative super-ring $R = R_0 + R_1$ is nothing but an associative $(\mathbb{Z}/2\mathbb{Z})$-graded ring. A $(\mathbb{Z}/2\mathbb{Z})$-graded ideal $I = I_0 + I_1$ of an associative super-ring $R$ is called a superideal of $R$. An associative super-ring $R$ is simple if it has no non-trivial superideals. Let $R$ be an associative super-ring with $1 \in R_0$ then $R$ is said to be a division super-ring if all nonzero homogeneous elements are invertible, i.e., every $0 \neq r_{\alpha} \in R_{\alpha}$ has an inverse $r_{\alpha}^{-1}$, necessarily in $R_{\alpha}$.

An associative $(\mathbb{Z}/2\mathbb{Z})$-graded $K$-algebra $A = A_0 + A_1$ is a finite dimensional central simple superalgebra over a field $K$, if $Z(A) \cap A_0 = K$, where $Z(A) = \{a \in A \mid ab = ba \ \forall b \in A\}$ is the center of $A$, and the only superideals of $A$ are $(0)$ and $A$ itself.

By [6, Theorem 3] finite dimensional central simple associative superalgebras over a field $K$ are isomorphic to $\text{End} V \cong M_n(D)$, where $D = D_0 + D_1$ is a finite dimensional associative division superalgebra over $K$, i.e., all nonzero elements of $D_{\alpha}$, $\alpha = 0, 1$, are invertible, and $V = V_0 + V_1$ is an $n$-dimensional $D$-superspace.
If \( D_1 = \{0\} \), the grading of \( M_n(D) \) is induced by that of \( V = V_0 + V_1 \), \( A = M_{p+q}(D) \), \( p = \dim_D V_0, q = \dim_D V_1 \), so \( p + q \) is a non-trivial decomposition of \( n \). While if \( D_1 \neq \{0\} \) then the grading of 
\( M_n(D) \) is given by \((M_n(D))_{\alpha} = M_n(D_{\alpha})\), \( \alpha = \bar{0}, \bar{1} \).

In [1] A. Elduque and O. Villa proved some results about superinvolutions over a field of characteristic not 2, which is not the case of this paper.

**Theorem 1.1** (Division Superalgebra Theorem). (See [6].) If \( D = D_0 + D_1 \) is a finite dimensional associative division superalgebra over a field \( K \) then exactly one of the following holds where throughout \( \mathcal{E} \) denotes a finite dimensional associative division algebra over \( K \).

1. \( D = D_0 = \mathcal{E}, \) and \( D_1 = \{0\} \).
2. \( D = \mathcal{E} \otimes_k K[u], u^2 = \lambda \in K^\times, D_0 = \mathcal{E} \otimes K 1, D_1 = \mathcal{E} \otimes K u. \)
3. \( D = \mathcal{E} \) or \( M_2(\mathcal{E}), u \in D \) such that \( u^2 = \lambda \in K^\times, D_0 = C_D(u), D_1 = S_D(u), \) where \( C_D(u) = \{d \in D \mid du = ud\}, S_D(u) = \{d \in D \mid du = -ud\}, \) moreover, in the second case, \( u = \left( \begin{array}{cc} 0 & 1 \\ \lambda & 0 \end{array} \right) \) and \( K[u] \) does not embed in \( \mathcal{E} \).

Following [5] we say that a division superalgebra \( D \) is even if \( Z(D) \cap D_1 = \{0\} \), where \( Z(D) \) is the center of \( D \), i.e., \( D \) is even if its form is (i) or (iii), and that \( D \) is odd if its form is (ii). Also, if \( A = M_n(D) \) is a finite dimensional central simple superalgebra over a field \( K \), then we say that \( A \) is even \( K \)-superalgebra if \( D \) is an even division superalgebra and \( A \) is odd \( K \)-superalgebra if \( D \) is an odd division superalgebra.

**2. Definitions and examples**

**Definition 1.** An anti-automorphism of an associative superalgebra \( A \) is a graded additive map \( \ast : A \to A \) such that

\[(a_\alpha b_\beta)^* = (-1)^{\alpha \beta} b_\beta^* a_\alpha^* .\]

If \( A \) is a finite dimensional central simple superalgebra over a field \( K \), and \( \ast \) is an anti-automorphism of order two on \( A \), that is

\[a^{**} = a \quad \forall a \in A,\]

then \( \ast \) is called a superinvolution on \( A \). Since \( K = Z(A) \cap A_\bar{0}, \) \( K^* = K, \) that is \( \alpha^* \in K \forall \alpha \in K, \) so we say that \( \ast \) is a superinvolution of the first kind if the restriction \( \ast|_K = id_K, \) and it is a superinvolution of the second kind if the restriction \( \ast|_K = \sigma, \) where \( \sigma \) is a Galois automorphism of order 2 on \( K. \)

If \( \ast \) is a superinvolution on a superalgebra \( A \), then we say that \((A, \ast)\) is simple if and only if the \( \ast \)-stable superideals of \( A \) are \( \{0\} \) and \( A \) itself.

**Definition 2.** Let \( A \) be any \( K \)-superalgebra, we define the map \( \varphi : A \to A \) by

\[a_\alpha^\varphi = (-1)^\alpha a_\alpha \quad \forall a_\alpha \in A_\alpha \text{ and } \forall \alpha = \bar{0}, \bar{1}.\]

This map \( \varphi \) is a superalgebra automorphism, called the sign automorphism, since

\[(a_\alpha b_\beta)^\varphi = (-1)^{\alpha + \beta} a_\alpha b_\beta = a_\alpha^\varphi b_\beta^\varphi \]

for all \( a_\alpha \in A_\alpha \) and \( b_\beta \in A_\beta. \) The automorphism \( \varphi \) has order 2, if \( \text{Char}(K) \neq 2 \) (unless \( A_1 = 0 \), and \( \varphi = id_A \) if \( \text{Char}(K) = 2. \)
Definition 3. If $R = R_0 + R_1$ is an associative super-ring, a (right) $R$-supermodule $M$ is a right $R$-module with a grading $M = M_0 + M_1$ as $R_0$-modules such that $m_α r_β \in M_{α+β}$ for any $m_α \in M_α$, $r_β \in R_β$, $α, β \in \mathbb{Z}_2$. An $R$-supermodule $M$ is simple if $MR \neq \{0\}$ and $M$ has no proper subsupermodule.

Following [6] we have the following definition of $R$-supermodule homomorphism.

Definition 4. Suppose $M$ and $N$ are $R$-supermodules. An $R$-supermodule homomorphism from $M$ into $N$ is an $R_0$-module homomorphism $h_γ : M → N$, $γ \in \mathbb{Z}_2$, such that $M_α h_γ \subseteq N_{α+γ}$ and

$$(m_α r_β) h_γ = (m_α h_γ) r_β \quad ∀ m_α \in M_α, \ r_β \in R_β, \ α, β \in \mathbb{Z}_2.$$

Definition 5. The opposite super-ring $R^o$ of the super-ring $R$ is defined to be $R^o = R$ as an additive group, with the multiplication given by

$$b_β \circ c_γ := (-1)^{βγ} c_γ b_β, \quad b_β \in R_β, \ c_γ \in R_γ.$$

Definition 6. Let $A = A_0 + A_1, B = B_0 + B_1$ be associative superalgebras. Then the graded tensor product

$$A \hat{⊗}_K B = [(A_0 \otimes B_0) \oplus (A_1 \otimes B_1)] \oplus [(A_0 \otimes B_1) \oplus (A_1 \otimes B_0)]$$

where the multiplication on $A \hat{⊗}_K B$ is induced by

$$(a_α \otimes b_β)(c_γ \otimes d_δ) = (-1)^{βγ} a_α c_γ \otimes b_β d_δ, \quad a_α \in A_α, \ c_γ \in A_γ, \ b_β \in B_β, \ d_δ \in B_δ.$$

If $A$ and $B$ are associative superalgebras, then $A \hat{⊗}_K B$ is an associative superalgebra.

So, if $A$ is a superalgebra then $A^o$ is just the opposite super-ring of $A$; one can easily show that if $A$ is a central simple associative superalgebra over a field $K$, then $A^o$ is also a central simple associative superalgebra over $K$, and by [5] $A \hat{⊗}_K A^o \cong M_n(K)$, where $n = \dim_K(A)$.

Examples (of associative superalgebras). (i) Let $K$ be a field of characteristic not 2, and let $λ, μ \in K \setminus \{0\}$. Then the quaternion algebra

$$A = K1 + Ku + Kv + Kuv,$$

where $u^2 = λ, v^2 = μ, \text{ and } uv = -vu$, is a central simple superalgebra $A = (λ, μ)$ with the grading

$$A_0 = K1 + Kuv, \quad A_1 = Ku + Kv.$$

(ii) Let $K$ be a field of characteristic 2, and let $λ \in K \setminus \{0, α^2 | α \in K\}$. Then for $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $w = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$$A = K[u] \oplus K[u]w$$

is a central simple division superalgebra over $K$ ($A$ is a quaternion algebra over $K$ of characteristic 2) with grading

$$A_0 = K[u], \quad A_1 = K[u]w.$$

(iii) The algebra of $p + q \times p + q$ matrices $M_{p+q}(D)$, where $D$ is a division algebra, can be viewed as an associative superalgebra by taking the diagonal components $M_p(D)$ and $M_q(D)$ as the even part.
and the off-diagonal components as the odd part; this is an example of simple associative superalgebra.

(iv) A superspace over $K$ is a left $K$-vector space which is $\mathbb{Z}_2$-graded $V = V_0 \oplus V_1$. The associative algebra $\text{End}_K V = [\text{End}_K V]_0 + [\text{End}_K V]_1$, where

$$[\text{End}_K V]_\alpha := \{a \in \text{End}_K V \mid v_\beta a \in v_{\alpha + \beta}\},$$

is an associative superalgebra. A symmetric superform on $V$ is a graded bilinear form

$$(.,.) : V \times V \to K, \quad V = V_0 \perp V_1,$$

which is symmetric on $V_0$ and skew-symmetric on $V_1$. The symmetric superform $(.,.)$ is nondegenerate if $(v_\alpha, v) = \{0\}$ implies that $v_\alpha = 0$.

One can easily check that a nondegenerate symmetric superform on a finite dimensional $V$ induces a superinvolution $\ast$ on $\text{End}_K V$ via

$$(v_\alpha a_\gamma, w_\beta) = (-1)^{\beta\gamma}(v_\alpha, w_\beta a_\gamma^\ast), \quad \text{for all } v_\alpha, w_\beta \in V.$$ 

**Definition 7.** Two finite dimensional central simple superalgebras $A$ and $B$ over a field $K$ are called similar ($A \sim B$) if there exist graded $K$-vector spaces $V = V_0 \oplus V_1, W = W_0 \oplus W_1$, such that $A \hat{\otimes}_K \text{End}_K V \cong B \hat{\otimes}_K \text{End}_K W$ as a $K$-superalgebra.

Similarity is obviously an equivalence relation. The set of similarity classes will be denoted by $\text{BW}(K)$ (the Brauer–Wall group of $K$). If $[A]$ denotes the class of $A$ in $\text{BW}(K)$ by using [5, Chap. 4, Theorem 2.3 (3)] the operation $[A][B] = [A \hat{\otimes}_K B]$ is well defined, and makes the set of similarity classes of finite dimensional central simple superalgebras over $K$ into a commutative group, $\text{BW}(K)$, where the class of the matrix algebras $M_{p+q}(K)$ is a neutral element for this product.

### 3. Existence of a superinvolution of the first kind

**Lemma 3.1.** If $A$ is an even central simple superalgebra over field $K$. Then the sign automorphism $\varphi$ is an inner automorphism. If $A$ is odd and $\text{Char}(K) \neq 2$ then $\varphi$ is not inner.

**Proof.** By [8, p. 438], $A = M_n(D)$, where $D$ is a finite dimensional even division superalgebra over $K$.

If $\text{Char}(K) = 2$ then $\varphi = \text{id}_A$, and hence its corresponds to conjugating by $u = I_n$, where $I_n$ is the $n \times n$ identity matrix.

Assume that $\text{Char}(K) \neq 2$, if $A = M_{p+q}(D)$ where $D$ is a finite dimensional central simple division algebra over $K$. Then $a^\varphi = u a u^{-1}$ where $u = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. If $A = M_n(D)$ where $D$ is a finite dimensional even division superalgebra with non-trivial grading over $K$. Then $a^\varphi = u I_n a u^{-1} I_n$ where $u$ is as defined in type (iii) in Theorem 1.1.

If $A$ is odd then $A = M_{n}(D)$ where $D = D_0 + D_0 u$, $u^2 = \lambda \in K^\times$, is an odd division superalgebra. If $b_\beta$ is an invertible element in $A_\beta$ such that $a^\varphi = b_\beta a_\alpha b_\beta^{-1}$ for all $a_\alpha \in A_\alpha$ then $u^\varphi = -u = b_\beta u b_\beta^{-1} = u$ since $u \in Z(A)$, a contradiction. \(\square\)

**Theorem 3.2** (Skolem–Noether Theorem). Let $B$ be a central simple superalgebra over the field $K$, and let $A$ be a finite dimensional simple subsuperalgebra over $K$ and containing it. Then any superalgebra homomorphism $f$ of $A$ into $B$ can be extended to an inner automorphism of $B$ if $B$ is even. If $B$ is odd then $f$ or $f\varphi$ can be extended to an inner automorphism but not both of them, where $\varphi$ is the sign automorphism.
Proof. Let \( E = B^\circ \otimes_K A \) where \( B^\circ \) is the opposite superalgebra of \( B \), then by [5, Theorem 2.3 (2)] \( E \) is a simple superalgebra over \( K \). Using the homomorphism \( f \) of \( A \) into \( B \), we make \( B \) into a right \( E \)-supermodule in two ways. In the first way, the action is \( x_{\gamma}.(b_\beta \otimes a_\alpha) = (-1)^{\beta\gamma} b_\beta x_{\gamma} a_\alpha \) and the second action is \( x_{\gamma}.(b_\beta \otimes a_\alpha) = (-1)^{\beta\gamma} b_\beta x_{\gamma} a_\alpha^f \), where \( f \) is the given superalgebra homomorphism of \( A \) into \( B \). Then \( B \) is a right \( E \)-supermodule under these two actions. By [6, Proposition 4], these supermodules are isomorphic. Hence there exists an isomorphism \( s_\delta \) such that
\[
(\gamma) x_{\gamma}.(b_\beta \otimes a_\alpha) = s_\delta(x_{\gamma}).(b_\beta \otimes a_\alpha^f)
\]
Therefore
\[
(-1)^{\beta\gamma} s_\delta(b_\beta x_{\gamma} a_\alpha) = (-1)^{\beta(\gamma + \delta)} b_\beta s_\delta(x_{\gamma}) a_\alpha^f
\]
and so
\[
s_\delta(b_\beta x_{\gamma} a_\alpha) = (-1)^{\beta\delta} b_\beta s_\delta(x_{\gamma}) a_\alpha^f.
\]
For \( b_\beta = 1 \), \( s_\delta(x_{\gamma} a_\alpha) = s_\delta(x_{\gamma}) a_\alpha^f \), so if \( x_{\gamma} = 1 \), then
\[
s_\delta(a_\alpha) = s_\delta(1) a_\alpha^f.
\]
Now, in (3.1) let \( a_\alpha = 1 \), then \( s_\delta(b_\beta x_{\gamma}) = (-1)^{\beta\delta} b_\beta s_\delta(x_{\gamma}) \) and so \( x_{\gamma} = 1 \), yields \( s_\delta(b_\beta) = (-1)^{\beta\delta} b_\beta s_\delta(1) \), but from (3.2),
\[
(s_\delta(1)) b_\beta^f = (-1)^{\beta\delta} b_\beta s_\delta(1)
\]
and therefore
\[
b_\beta^f = (-1)^{\beta\delta} (s_\delta(1))^{-1} b_\beta s_\delta(1).
\]
For \( \delta = 0 \)
\[
b_\beta^f = (s_\delta(1))^{-1} b_\beta s_\delta(1).
\]
For \( \delta = 1 \)
\[
b_\beta^f = (s_\delta(1))^{-1} b_\beta s_\delta(1).
\]
For \( B \) even, \( \varphi \) is an inner automorphism but not for odd \( B \). Therefore \( f \) can be extended to an inner automorphism on \( B \) if it is even. If \( B \) is odd then \( f \) or \( f \varphi \) is inner but not both of them. \( \square \)

In [6, Theorem 3] Michel Racine proved that finite dimensional associative central simple superalgebras \( A = M_n(\mathcal{D}) \) over a field \( K \) are primitive superalgebras, and then he proved in [6, Theorem 7] that a primitive superalgebra \( A = M_n(\mathcal{D}) \) has a superinvolution if and only if \( \mathcal{D} \) has. Thus we have the following result.

**Theorem 3.3.** A finite dimensional associative central simple superalgebra \( A = M_n(\mathcal{D}) \) over a field \( K \) has a superinvolution * if and only if \( \mathcal{D} \) has.

If \( A = M_n(\mathcal{D}) \) is a finite dimensional central simple superalgebra over a field \( K \), where \( \mathcal{D} \) is a finite dimensional division superalgebra with non-trivial grading over \( K \), that is \( \mathcal{D}_1 \neq [0] \), then by Theorem 3.3, it is enough to classify the existence of superinvolutions on \( \mathcal{D} \) to ascertain the existence of superinvolutions on \( A \).
Now for a finite dimensional division superalgebra $\mathcal{D}$ over $K$, we have the following result.

**Theorem 3.4.**

(1) Let $\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_0 u$ be an odd division superalgebra. If $K$ is a field of characteristic not 2 then $\mathcal{D}$ doesn’t admit a superinvolution of the first kind.

(2) If $\mathcal{D}$ is an even division superalgebra with non-trivial grading over any field $K$ of characteristic not 2 then $\mathcal{D}$ doesn’t admit a superinvolution of the first kind.

**Proof.** (1) Let $*$ be a superinvolution of $\mathcal{D}$, then

$$(u^2)^* = -(u^*)^2 \quad \text{implies that} \quad \lambda^* = -\lambda \in K.$$  

So $*$ is a superinvolution of the second kind.

(2) Now for an even $\mathcal{D}$ we give a proof by contradiction.

Assume that $\mathcal{D}$ admits a superinvolution $*$ of the first kind and let $\sim = *|_{\mathcal{D}_0}$. By [6, Proposition 10] $\mathcal{D}_1$ contains a $0 \neq v = v^*$ so let $\phi: \mathcal{D} \to \mathcal{D}$ be defined by $x^\phi = v xv^{-1}$. If $\sim$ is an involution on $\mathcal{D}_0$ of the first kind then $\sim \phi$ is an involution on $\mathcal{D}_0$ of the second kind and vice versa.

Assume that $\sim$ is of the first kind, we have

$$\mathcal{H}(K(u), \sim \phi) = \{ x \in K(u) \mid \sim \phi x = x \} = K,$$

where $u$ is as defined in type (iii) in Theorem 1.1. Let $z = \frac{u \otimes u}{\lambda} \in \mathcal{D}_0 \otimes_K K(u)$, and let $e = \frac{1 - z}{2}$, then $(\mathcal{D}_0 \otimes_K K(u))e$ is a $(\sim \otimes 1)$-stable proper ideal in $\mathcal{D}_0 \otimes_K K(u)$. Therefore $(\mathcal{D} \otimes_K K(u), \sim \otimes 1)$ is as in [6, Theorem 12]. Now $\mathcal{D} \otimes_K K(u) \cong M_n(C')$, where $C'$ is a central simple division algebra over $K(u)$, the grading on $M_n(C')$ is not inherited from $C'$, because if the grading is inherited from $C'$ then $Z(M_n(C'))$ is a field and equal to $Z(\mathcal{D}_0 \otimes_K K(u)) = K(u) \otimes_K K(u)$, a contradiction. So $\mathcal{D} \otimes_K K(u) \cong M_{p+q}(C')$, where $n = p + q$, but

$$\dim_{K(u)}(\mathcal{D}_0 \otimes_K K(u)) = \dim_{K(u)}(\mathcal{D}_1 \otimes_K K(u)),$$

hence $p = q$, and therefore $\mathcal{D} \otimes_K K(u) \cong M_{2p}(C')$, which implies that

$$\mathcal{D}_0 \otimes_K K(u) \cong M_p(C') \oplus M_p(C'),$$

by [6, Proposition 14], $\sim \otimes 1$ restricts to an orthogonal involution on one of the summands of $\mathcal{D}_0 \otimes_K K(u)$ and to a symplectic involution on the other summand. Thus

$$\dim_{K(u)} \mathcal{H}(\mathcal{D}_0 \otimes_K K(u), \sim \otimes 1) = \dim_{K(u)} \mathcal{H}(M_p(C'), \sim \otimes 1) + \dim_{K(u)} S(M_p(C'), \sim \otimes 1)$$

$$= \dim_{K(u)} M_p(C')$$

$$= \dim_{K(u)} \mathcal{D}_0$$

where $S(M_p(C'), \sim \otimes 1) = \{ x \in M_p(C') \mid x^{\sim \otimes 1} = -x \}$. But this is impossible since

$$\dim_{K(u)} \mathcal{H}(\mathcal{D}_0 \otimes_K K(u), \sim \otimes 1) = \dim_{K(u)} \mathcal{H}(\mathcal{D}_0, \sim) \otimes_K K(u)$$

$$= \dim_{K} \mathcal{H}(\mathcal{D}_0, \sim)$$

$$= 2 \dim_{K(u)} \mathcal{H}(\mathcal{D}_0, \sim).$$
Now if ~ is of the second kind then
\[ a + bν^{-1} \mapsto \tilde{a}^\phi + \tilde{b}ν^{-1}, \quad a, b \in D_0 \]
is another superinvolution on D whose restriction to \( D_0 \) is of the first kind and will lead to the contradiction above. \( \square \)

Thus, if \( K \) is a field of characteristic not 2 and \( A \) is a \( K \)-superalgebra with a superinvolution of the first kind (say \( * \)) then by Theorem 3.4, \( A = M_{p+q}(C) \) where \( C \) is a finite dimensional division algebra over \( K \). Moreover, if \((A_0, *|_{A_0})\) is simple then by [6, Proposition 13] \((A, \star)\) is isomorphic to \( M_{2p}(C) \) with the superinvolution \( \star \) given by

\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \star = \left( \begin{array}{cc} \tilde{a} & -\tilde{b} \\ \tilde{c} & \tilde{a} \end{array} \right),
\]

for \( a, b, c, d \in M_p(C) \), but if \((A_0, *|_{A_0})\) is not simple then by [6, Proposition 14] \( A_0 = B_1 \oplus B_2 \), where \( B_1 = M_p(C) \) and \( B_2 = M_q(C) \) and one of \((B_1, *|_{B_1})\) is of orthogonal type and the other of symplectic type (i.e., at least one of \( p \) and \( q \) is even). Therefore, we have the following result, but first of all we recall Albert’s Lemma in the algebra case: if \( \mathcal{A} \) is a finite dimensional central simple algebra over a field \( K \). Then \( \mathcal{A} \) admits an involution of the first kind if and only if \( \mathcal{A} \cong \mathcal{A}^\circ \).

**Theorem 3.5.**

(1) Let \( K \) be a field of characteristic not 2, and let \( C \) be a finite dimensional central division algebra over \( K \). Let \( A = M_{p+q}(C) \) be a \( K \)-superalgebra, where \( p \) or \( q \) is even if \( p \neq q \). Then \( A \) has a superinvolution of the first kind if and only if \( A \) is of order 2 in the Brauer–Wall group \( BW(K) \).

(2) Let \( K \) be a field of characteristic 2, and let \( C \) be a finite dimensional central division algebra over \( K \). Let \( A = M_{p+q}(C) \) be a \( K \)-superalgebra. Then \( A \) has a superinvolution of the first kind if and only if \( A \) is of order 2 in the Brauer–Wall group \( BW(K) \).

**Proof.** (1) If \( A \) has a superinvolution of the first kind then \( A \cong A^\circ \), therefore \( A \) is of order 2 in the \( BW(K) \).

Conversely, if \( \mathcal{A} \) is of order 2 in the Brauer–Wall group \( BW(K) \), then \( C \) is of order 2 in the Brauer group \( Br(K) \) and hence by Albert’s Theorem \( C \) has an involution of the first kind, so if \( p = q \) then by [6, Proposition 13] \( A \) has a superinvolution of the first kind, and if \( p \neq q \) then by [6, Proposition 14] \( A \) has a superinvolution of the first kind.

(2) If \( A \) is of order 2 in the Brauer–Wall group \( BW(K) \), then \( C \) is of order 2 in the Brauer group \( Br(K) \), and hence by Albert’s Theorem \( C \) has an involution of the first kind (say \( \star \)). Therefore, \( t \otimes \star \), where \( t \) is the transpose involution on \( M_{p+q}(K) \), is a superinvolution on

\[
A \cong M_{p+q}(K) \otimes_K C.
\]

Conversely, if \( A \) has a superinvolution of the first kind, then clearly, \( A \) is of order 2 in the Brauer–Wall group \( BW(K) \). \( \square \)

Moreover, we will give an example of a superalgebra \( A \) of order 2 in the Brauer–Wall group \( BW(K) \) (i.e., \( A \cong A^\circ \)) that doesn’t have a superinvolution of the first kind.

**Example.** Let \( K \) be any field of characteristic not 2 such that \( i = \sqrt{-1} \in K \). Let \( \lambda, \mu \in K \setminus \{0\} \) and let \( A = (\lambda, \mu) \) be the quaternion algebra on two generators \( u, v \) with defining relations: \( u^2 = \lambda, \ v^2 = \mu, \ uv = -vu \), as defined in Example (i). Then

\[
A = K \oplus Ku \oplus Kv \oplus Kuv \quad \text{where } A_0 = K \oplus Ku, \ A_1 = Kv \oplus Kuv,
\]
is a division superalgebra with basis \{1, u, v, uv\}. By Theorem 3.4, A doesn’t have a superinvolution of the first kind but it is of order 2 in the Brauer–Wall group \(\text{BW}(K)\). To see this define the \(K\)-linear map \(\ast: A \to A\) as follows: \(\alpha^x = x\) \(\forall x \in K\); \(u^x = u\); \(v^x = iv\); \((uv)^x = ivu = -iv\) then \(\ast\) is a \(K\)-anti-automorphism on \(A\), which implies that \(A\) is of order 2 in the Brauer–Wall group \(\text{BW}(K)\).

Let \(D = D_0 + D_0v\) be an even division superalgebra with a non-trivial grading (i.e., \(v \neq 0\) over the field \(K\) of characteristic 2, a \(K\)-anti-automorphism \(J\) is simply an isomorphism \(\cong\) (or: the opposite). Fix such a \(J\). Then we may assume \(x^J = x \forall x \in D_0\). For if not, we can define another \(K\)-anti-automorphism \(I\) on \(D\) such that \(x^I = x \forall x \in D_0\). To show this assume that \(J^2|_{D_0} \neq id_{D_0}\) but since

\[
J|_{D_0}: D_0 \cong D_0^0,
\]

so \(x^J = gxg^{-1}\), where \(g \in D_0\), and \(g^Jg^{-1} = 1 \quad[7, \text{Lemma 8.2}]. \) Let \(\alpha = (1 + \gamma)^{-1} (\gamma \neq -1)\). An easy computation shows that \(x^J = \alpha x^J \alpha^{-1} \forall x \in D\) is another graded \(K\)-anti-automorphism on \(D\), and \(x^J = x \forall x \in D_0 \quad[7, \text{Lemma 8.2}]. \) Therefore, we may fix a \(K\)-anti-automorphism \(J\) on \(D\) such that \(x^J = x \forall x \in D_0\).

**Lemma 3.6.** If an even division superalgebra \(D\) with non-trivial grading over the field \(K\) of characteristic 2, admits a \(K\)-anti-automorphism \(J\) such that \(x^J = x \forall x \in D_0\) then \(J^2\) is an inner automorphism, and \(x^J = bx^{-1}\) for \(b \in Z(D_0)\).

**Proof.** The map \(J^2\) is a \(K\)-automorphism on \(D\), hence an inner automorphism \(x^J = bx^{-1}\) for a suitable invertible element \(b = b_0 + b_1 \in D\). If \(\text{Char}(K) = 2\) then \(u^J = u = (b_0 + b_1)u(b_0 + b_1)^{-1} = (ub_0 + (u + 1)b_1)(b_0 + b_1)^{-1} = u\) implies that \((u(b_0 + b_1) + b_0(b_0 + b_1)^{-1}) = u\) and so \(u + b_1(b_0 + b_1)^{-1} = 0\) implies that \(b_1 = 0\). Thus \(b = b_0 \in D_0\). But \(x^J = x \forall x \in D_0\), so \(b \in Z(D_0)\).

**Lemma 3.7.** If \(D\) is as in the lemma above and if \(D_0^J \cong D\) and \(D_0^J \cong D\) such that \(J^2|_{D_0} = J^2|_{D_0} = id_{D_0}\), where \(I\) and \(J\) are \(K\)-anti-automorphisms on \(D\). Then there exists \(a_\alpha \in D_\alpha\) such that \(x^I = a_\alpha x^I a_\alpha^{-1} \forall x \in D\).

**Proof.** (\(\text{Char} K = 2\)). Since \(u^I \in Z(D_0) = (K(u) = K(u)\) then \(u^I = \alpha + \beta u\), also \(v^I = dv \in D_0\) and \(v^I = d_1\), this implies that \(v^Iv^I = \alpha + \beta uv + uv\) \(\alpha + \beta(1 + u) = \alpha + \beta + \beta u = \beta + u^I\), hence \(v^Iv^I = (\alpha + u^I) = (\alpha + u + u = u)\) we have \(\alpha^I + \alpha + 0 = \alpha\) and \(\alpha^I = \alpha \in K\). If \(\alpha = 0\) then \(u^I = u\); if not then replace \(u\) by \(\frac{1}{\alpha} u\) we get \(u^I = 1 + u\). Therefore, we have two cases: \(u^I = u\) or \(u^I = 1 + u\), let \(u^I = \gamma + u\) for some \(\gamma \in K\). Now, for \(u^I = 1 + u\) and by using the Skolem–Noether Theorem we have

\[
u^I = \gamma + u = au^Ia^{-1} = a(1 + u)a^{-1}
\]

\[
= \alpha_{0\alpha} + a_{1\alpha}(1 + u)(\alpha_{0\alpha} + a_{1\alpha})^{-1}
\]

\[
= 1 + (\alpha_{0\alpha} + a_{1\alpha})u(\alpha_{0\alpha} + a_{1\alpha})^{-1}
\]

\[
= 1 + (u(\alpha_{0\alpha} + a_{1\alpha}) + a_{1\alpha}(\alpha_{0\alpha} + a_{1\alpha})^{-1}
\]

\[
= 1 + a_{1\alpha}(\alpha_{0\alpha} + a_{1\alpha})^{-1}
\]

so \(\gamma + 1 = a_{1\alpha}(\alpha_{0\alpha} + a_{1\alpha})^{-1}\) and therefore \((\gamma + 1)(\alpha_{0\alpha} + a_{1\alpha}) = a_{1\alpha}\) and so \(\gamma + 1) = \gamma a_{1\alpha}\) which implies that \(\gamma a_{1\alpha} = 0\) so \(\gamma = 0\) or \(a_{1\alpha} = 0\) but if \(\gamma = 0\) then \(a_{0\alpha} = 0\). Therefore \(a = a_\alpha \in D_\alpha\).
For $u^J = u$, again by using the Skolem–Noether Theorem, we have

$$u^J = \gamma + u = au^Ja^{-1} = au^{-1}$$

$$= (a_0 + a_1)u(a_0 + a_1)^{-1}$$

$$= (ua_0 + (u + 1)a_1)(a_0 + a_1)^{-1}$$

$$= (u(a_0 + a_1) + a_1)(a_0 + a_1)^{-1}.$$  

So, $\gamma + u = u + a_1(a_0 + a_1)^{-1}$ and therefore $\gamma(a_0 + a_1) = a_1$ and so $(\gamma + 1)a_1 = \gamma a_0$ which implies that $\gamma a_0 = 0$ so $\gamma = 0$ or $a_0 = 0$ but if $\gamma = 0$ then $a_1 = 0$. Therefore $a = a_0 \in D_\alpha$. \qed

**Lemma 3.8.** Let $b \in D_0$ be as in Lemma 3.6. Then:

(i) $bb^J = b^Jb \in K^\times$.

(ii) $bb^J$ does not depend on the choice of $J$ and $b$.

**Proof.** (i) The equation $x_{a}^2 = bx_\alpha b^{-1}$ implies

$$x_{a}^3 = (x_{a}^2)^J = (bx_\alpha b^{-1})^J = b^{-J}x_{a}^J b^J$$

$$= (x_{\alpha}^J)^2$$

$$= bx_\alpha b^{-1}.$$

$\Rightarrow bx_\alpha b^{-1} = b^{-J}x_\alpha b^J \Rightarrow x_{\alpha}^J = b^{-1}b^{-J}x_\alpha b^J b$. Hence $b^Jb \in K$. Therefore $(b^Jb)^Jbb^J = b(b^Jb)^Jb^J = b(b^Jb)bb^J = (bb^J)(bb^J) \Rightarrow b^Jb = bb^J$.

(ii) Let $I$ be another $K$-anti-automorphism on $D$ such that $x^I = x \forall x \in D_0$ then, by Lemma 3.7, there exists $a_\alpha \in D_\alpha$ such that $x^I = a_\alpha x^I a_\alpha^{-1} \forall x \in D$.

For $\alpha = 0$:

$$x_{\alpha}^2 = a_0(a_0x_\alpha a_0^{-1})^Ja_0^{-1}$$

$$= a_0a_0^{-J}bx_\alpha b^{-1}a_0^Ja_0^{-1}.$$

**Claim.** $(a_0a_0^{-J}b)(a_0a_0^{-J}b)^I = bb^J$.

**Proof of the claim.**

$$(a_0a_0^{-J}b)(a_0a_0^{-J}b)^I = a_0a_0^{-J}ba_0(a_0a_0^{-J}b)^Ja_0^{-1}$$

$$= a_0a_0^{-J}ba_0b^{-1}a_0^{-1}b^{-1}a_0^Ja_0^{-1}$$

$$= b^Jb. \quad \square$$
For $\alpha = \bar{1}$:

$$x_{\beta}^2 = a_1(a_1 x_{\beta} a_1^{-1}) J a_1^{-1}$$

$$= (-1)^{(1+\beta)} (-1)^{\beta} a_1(a_1^{-1}) J bx_{\beta} b^{-1} a_1^{-1} a_1^{-1}$$

$$= -a_1(a_1^{-1}) J bx_{\beta} b^{-1} a_1^{-1}.$$ 

Since $(a_1 a_1^{-1}) J 1 = 1 = -(a_1^{-1}) J a_1^{-1} = 1$, we have $(a_1^{-1}) J = -(a_1 J)^{-1}$ and therefore

$$x_{\beta}^2 = a_1(a_1 J)^{-1} bx_{\beta} b^{-1} a_1^{-1} a_1^{-1}.$$ 

Claim. $(a_1(a_1 J)^{-1} b)(a_1(a_1 J)^{-1} b)^ J = bb J.$

Proof of the claim.

$$(a_1(a_1 J)^{-1} b)(a_1(a_1 J)^{-1} b)^ J = a_1(a_1 J)^{-1} b a_1(a_1 J)^{-1} b)^ J a_1^{-1}$$

$$= a_1(a_1 J)^{-1} b a_1(-1)^{b J} (-1)^{b J} a_1^{-1} a_1^{-1} a_1^{-1}$$

$$= a_1(a_1 J)^{-1} b a_1 b a_1^{-1} b a_1^{-1} a_1^{-1}$$

$$= b J b$$

$$= bb J. \quad \square$$ 

Theorem 3.9. Let $D = D_0 + D_0 v$ be an even central division superalgebra over a field $K$ of characteristic 2 such that $v \neq 0$, and let $J : D \rightarrow D$ be any $K$-anti-automorphism on $D$ such that $x J^2 = b x b^{-1}$ for all $x \in D$ where $b \in Z(D_0)$. Then $D$ has a superinvolution of the first kind if and only if

$$bb J \in N(K^\times) = \{ a^2 \mid \alpha \in K^\times \}.$$ 

Proof. If $bb J = \alpha^2$, where $\alpha \in K^\times$, then $(\frac{b J b}{\alpha} J)^ J = 1$, therefore we may assume that $bb J = 1$. If $b = -1$, then we are finished since $J$ is a superinvolution of the first kind. Otherwise a trivial computation shows that

$$I : D \rightarrow D, \quad x \mapsto (1 + b)^{-1} x J (1 + b)$$

is a superinvolution of the first kind on $D$.

Conversely, if $*$ is a superinvolution of the first kind on $D$ then choose $b = 1. \quad \square$

Theorem 3.10. If $D = D_0 + D_0 v$ is a non-trivial even central division superalgebra over a field $K$ of characteristic 2 such that $D \cong D^\circ$ then $D$ has a superinvolution of the first kind.

Proof. Since $D \cong D^\circ$, let $J : D \rightarrow D$ be any $K$-anti-automorphism on $D$ such that $x J^2 = b x b^{-1}$ for all $x \in D$ where $b \in Z(D_0)$. Also, since $\text{Char}(K) = 2$, $(xy)^ J = y J x J$ for all $x, y \in D$, thus, $J$ is a $K$-anti-automorphism on the central simple algebra $D$, which means that $D$ is of order 2 in the Brauer group $\text{Br}(K)$, therefore, by Albert’s Theorem, $D$ has an involution of the first kind which implies that $bb J \in N(K^\times) = \{ a^2 \mid \alpha \in K \}$, and so by Theorem 3.9, $D$ has a superinvolution of the first kind. \square
Thus, we have the following result:

**Theorem 3.11.** Let $D = D_{\overline{0}} + D_{\overline{0}}v$ be a non-trivial even central division superalgebra over a field $K$ of characteristic 2 then $D$ has a superinvolution of the first kind if and only if $D$ is of order 2 in the Brauer–Wall group $BW(K)$.

**Theorem 3.12.** Let $D = D_{\overline{0}} + D_{\overline{0}}v$, where $v \in Z(D)$ be a non-trivial odd central division superalgebra over the field $K$ of characteristic 2. Then $D$ has a superinvolution of the first kind if and only if $D$ is of order 2 in the Brauer–Wall group $BW(K)$.

**Proof.** If $D$ is of order 2 in the Brauer–Wall group $BW(K)$, then $D_{\overline{0}}$ is of order 2 in the Brauer group $Br(K)$, and hence by Albert’s Theorem $D_{\overline{0}}$ has an involution (say $J$) of the first kind. Now, let $*: D \to D$ be defined by $(a+bv)^* = a^J + b^Jv$, one can easily check that $*$ is a superinvolution of the first kind on $D$.

Conversely, if $D$ has a superinvolution of the first kind, then $D \cong D^\circ$ which means that $D$ is of order 2 in the Brauer–Wall group $BW(K)$. □

Theorems 3.2 and 3.4 have been proved in my thesis. (See [2, Theorem 2.1.5, Lemma 2.1.8].)

In [3, Theorem 3.3] I proved that an even division superalgebra has a pseudo-superinvolution of the first kind if and only if it is of order 2 in the Brauer–Wall group. Also in [3, Theorem 3.4] I proved that an odd division superalgebra $D$ has a pseudo-superinvolution of the first kind if and only if $\sqrt{-1} \in D$ and it is of order 2 in the Brauer–Wall group. This result is also proved in [1, Theorem 27]. Finally, some results about existence of superinvolution of the second kind have been introduced in [1] and [4].

**References**