FIXED POINT RESULTS ON COMPLETE G-METRIC SPACES

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Abstract

In this paper several fixed point theorems for a class of mappings defined on a complete G-metric space are proved. In the same time we show that if the G-metric space \((X, G)\) is symmetric then the existence and uniqueness of these fixed point results follows from the Hardy–Rogers theorem in the induced usual metric space \((X, d_G)\). We also prove fixed point results for mapping on a G-metric space \((X, G)\) by using the Hardy–Rogers theorem where \((X, G)\) need not be symmetric.

1. Introduction

During the sixties the notion of 2-metric space introduced by Gahler ([5], [6]) as a generalization of usual notion of the metric space \((X, d)\). But different authors proved that there is no relation between these two functions, for instance HA et al. in [7] showed that 2-metric need not be continuous function, further there is no easy relationship between results obtained in the two settings.

These consideration led Bapure Dhage in 1984 in his Ph.D. thesis to introduced a new class of generalized metric space called D-metric spaces ([1], [2]) as a generalization of usual notion of metric space.

In a subsequent series of papers Dhage attempted to develop topological structures in such spaces, by presenting various concepts of open balls in D-metric space (see [2], [3], [4]). He claimed that D-metrics provide a

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generalization of ordinary metric functions and went on to present several fixed point results. Subsequently these work have been the basis for over 40 papers by Dhage and other authors.

But in 2003 Zead Mustafa and Brailey Sims have demonstrated ([9]) that most of the claims concerning the fundamental topological structure of $D$-metric space are incorrect, for instance $D$-metric need not be a continuous function of its variables, despite Dhage’s attempts to construct such a topology, $D$-convergence of a sequence $(x_n)$ to $x$, in the sense that $D(x_m, x_n, x) \to 0$ as $n, m \to \infty$, need not correspond to convergence in any topology, also in 2005, S. V. R. Naidu, K. P. R. Rao, and N. S. Nivasa, showed that the various concepts of open balls in $D$-metric space are not valid, they give examples of complete $D$-metric spaces in which there are convergent sequences having two limits, also an example of a $D$-metric space in which $D$-metric convergent does not define a topology (see [13], [14]). Alternatively Zead Mustafa and Brailey Sims introduced more appropriate notion of generalized metric space called $G$-metric spaces (see [11]) as follows.

2. Preliminaries

Throughout this paper we denote $\mathbb{R}^+$ the set of all positive real numbers.

**Definition 1** (see [11]). Let $X$ be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$, be a function satisfying the following properties:

1. $G(x, y, z) = 0$ if $x = y = z$,
2. $0 < G(x, x, y)$ for all $x, y \in X$, with $x \neq y$,
3. $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$,
4. $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, (symmetry in all three variables), and
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the function $G$ is called *generalized metric*, or, more specifically *$G$-metric* on $X$, and the pair $(X, G)$ is called a *$G$-metric space*.

**Example 1** (see [11]). Let $(X, d)$ be a usual metric space, and define $G_s$ and $G_m$ on $X \times X \times X$ to $\mathbb{R}^+$, by

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z),$$

and

$$G_m(x, y, z) = \max \{ d(x, y), d(y, z), d(x, z) \}$$

for all $x, y, z \in X$, then $(X, G_s)$ and $(X, G_m)$ are $G$-metric spaces.
**Example 2** (see [10]). Let $X = \mathbb{R} \setminus \{0\}$. Define $G : X \times X \times X \to \mathbb{R}^+$, by
\[
G(x, y, z) = \begin{cases} 
1 + |x - y| + |y - z| + |x - z|, & \text{otherwise} \\
|x - y| + |y - z| + |x - z|, & \text{if } x, y, z \text{ all have the same sign.}
\end{cases}
\]

Then $(X, G)$ is a $G$-metric space.

**Definition 2** (see [11]). Let $(X, G)$ be a $G$-metric space, let $(x_n)$ be sequence of points of $X$, a point $x \in X$ is said to be the *limit* of the sequence $(x_n)$ if $\lim_{n,m \to \infty} G(x, x_n, x_m) = 0$, and we say that the sequence $(x_n)$ is $G$-convergent to $x$.

Thus, that if $x_n \xrightarrow{(G)} 0$, in a $G$-metric space $(X, G)$, then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$, (through this paper we mean by $\mathbb{N}$ the set of all natural numbers).

**Proposition 1** (see [11]). Let $(X, G)$ be a $G$-metric space. Then the following are equivalent.

1. $(x_n)$ is $G$-convergent to $x$.
2. $G(x_n, x_n, x) \to 0$, as $n \to \infty$.
3. $G(x_n, x, x) \to 0$, as $n \to \infty$.
4. $G(x_n, x, x) \to 0$, as $n, m \to \infty$.

**Definition 3** (see [11]). Let $(X, G)$ be a $G$-metric space, a sequence $(x_n)$ is called $G$-Cauchy if given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$ that is if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

**Proposition 2** (see [11]). If $(X, G)$ is a $G$-metric space, then the following are equivalent.

1. The sequence $(x_n)$ is $G$-Cauchy.
2. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

**Definition 4** (see [11]). Let $(X, G)$ and $(X', G')$ be two $G$-metric spaces, and let $f : (X, G) \to (X', G')$ be a function, then $f$ is said to be $G$-continuous at a point $a \in X$ if and only if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function $f$ is $G$-continuous on $X$ if and only if it is $G$-continuous at all $a \in X$.

**Proposition 3** (see [11]). Let $(X, G)$ and $(X', G')$ be two $G$-metric spaces. Then a function $f : X \to X'$ is $G$-continuous at a point $x \in X$ if and only if it is $G$-sequentially continuous at $x$; that is, whenever $(x_n)$ is $G$-convergent to $x$ we have $(f(x_n))$ is $G$-convergent to $f(x)$.
Definition 5 (see [11]). A $G$-metric space $(X, G)$ is called symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 4 (see [11]). Let $(X, G)$ be a $G$-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Proposition 5 (see [9]). Every $G$-metric space $(X, G)$ will define a metric space $(X, d_G)$ by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X.$$  

Note that if $(X, G)$ is a symmetric $G$-metric space, then

$$(2.1) \quad d_G(x, y) = 2G(x, y, y), \quad \forall x, y \in X.$$  

However, if $(X, G)$ is not symmetric, then it holds by the $G$-metric properties that

$$(2.2) \quad \frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \quad \forall x, y \in X$$  

and that in general these inequalities cannot be improved.

Definition 6 (see [11]). A $G$-metric space $(X, G)$ is said to be $G$-complete (or complete $G$-metric) if every $G$-Cauchy sequence in $(X, G)$ is $G$-convergent in $(X, G)$.

Proposition 6 (see [11]). A $G$-metric space $(X, G)$ is $G$-complete if and only if $(X, d_G)$ is a complete metric space.

The following fixed point theorem in usual metric space are proved from Hardy, G. E. and Rogers, T. D.

Theorem 2.1 (Hardy, see [8]). Let $(X, d)$ be a complete metric space, and $T$ be a function mapping $X$ into itself, suppose there exist nonnegative constants $a_i$ satisfying $\sum_{i=1}^{5} a_i < 1$ such that, the following condition satisfied,

$$(2.3) \quad d(T(x), T(y)) \leq a_1d(x, y) + a_2d(x, T(x)) + a_3d(y, T(y))$$

$$+ a_4d(x, T(y)) + a_5d(y, T(x)), \quad \forall x, y \in X.$$  

Then, $T$ has a unique fixed point (i.e., there exists $u \in X; Tu = u$).
3. Main Results

Here we start our work with the following theorem.

**Theorem 3.1.** Let \((X, G)\) be a complete \(G\)-metric spaces and let \(T : X \to X\) be a mapping which satisfies the following condition, for all \(x, y, z \in X\).

\[
G(T(x), T(y), T(z)) \leq \max \left\{ aG(x, y, z), b \left[ G(x, T(x), T(z)) + 2G(y, T(y), T(z)) \right], \\
b \left[ G(x, T(y), T(z)) + G(y, T(y), T(z)) + G(y, T(x), T(z)) \right] \right\},
\]

where \(0 \leq a < 1\), and \(0 \leq b < \frac{1}{3}\). Then \(T\) has a unique fixed point, say \(u\), and \(T\) is \(G\)-continuous at \(u\).

**Proof.** Suppose that \(T\) satisfies the condition (3.1), then for all \(x, y \in X\), we have

\[
G(T(x), T(y), T(z)) \leq \max \left\{ aG(x, y, z), b \left[ G(x, T(x), T(y)) + 2G(y, T(y), T(z)) \right], \\
b \left[ G(x, T(y), T(z)) + G(y, T(y), T(z)) + G(y, T(x), T(z)) \right] \right\},
\]

and

\[
G(T(y), T(x), T(z)) \leq \max \left\{ aG(y, x, z), b \left[ 2G(x, T(x), T(z)) + G(y, T(y), T(z)) \right], \\
b \left[ G(x, T(y), T(z)) + G(y, T(y), T(z)) + G(y, T(x), T(z)) \right] \right\}.
\]

Suppose that \((X, G)\) is symmetric. Then by definition of metric \((X, d_G)\) and the equations (2.1), (3.2) we get the following:

\[
d_G(T(x), T(y)) = 2G(T(x), T(y), T(z)) \leq 2 \max \left\{ \frac{a}{2} d_G(x, y), b \left[ \frac{1}{2} d_G(x, T(x)) + d_G(y, T(y)) \right], \\
\frac{b}{2} d_G(x, T(y)) + d_G(y, T(y)) + d_G(y, T(x)) \right\}.
\]
for all $x, y \in X$. We see that we have different cases and in each one the sums of all coefficients is less 1, as well as each case will be a special case of condition (2.3) of theorem (2.1), so the existence and uniqueness of the fixed point follow from theorem (2.1) in metric space $(X, d_G)$.

However, if $(X, G)$ is not symmetric, then by definition of metric $(X, d_G)$ and the equations (2.2), (3.2) and (3.3) we get

$$d_G(T(x), T(y)) = G(T(x), T(y), T(x)) + G(T(y), T(x), T(x))$$

$$\leq \max \left\{ \frac{2a}{3} d_G(x, y), b \left[ \frac{2}{3} d_G(x, T(x)) + \frac{4}{3} d_G(y, T(y)) \right] \right\}$$

$$+ \max \left\{ \frac{2a}{3} d_G(x, y), b \left[ \frac{4}{3} d_G(x, T(x)) + \frac{2}{3} d_G(y, T(y)) \right] \right\}$$

for all $x, y \in X$.

We see that the metric condition in equation (3.5) will gives no information about this map in the metric space $(X, d_G)$, since the sums of the coefficient in each possible case in (3.5) are not less than 1, so, we cannot apply theorem (2.1), but this can be proved by $G$-metric properties as follows.

Let $x_0 \in X$, be an arbitrary point and define the sequence $(x_n)$, by $x_n = T^n(x_0)$, then using (3.1) we get

$$G(x_n, x_{n+1}, x_{n+1})$$

$$\leq \max \left\{ \frac{aG(x_{n-1}, x_n, x_n),}{b \left[ G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}) \right]}, \frac{b \left[ G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}) \right]}{b} \right\}.$$
Then, equation (3.6) becomes,

\[
G(x_n, x_{n+1}, x_{n+1}) 
\]

\[
\leq \max \left\{ aG(x_{n-1}, x_n, x_n), b \left[ G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}) \right] \right\}.
\]

Hence, we have two cases

**Case (1):** If

\[
\max \left\{ aG(x_{n-1}, x_n, x_n), b \left[ G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}) \right] \right\} = b \left[ G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}) \right],
\]

then equation (3.7) becomes

\[
G(x_n, x_{n+1}, x_{n+1}) \leq \frac{b}{1 - 2b} G(x_{n-1}, x_n, x_n).
\]

**Case (2):** If

\[
\max \left\{ aG(x_{n-1}, x_n, x_n), b \left[ G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}) \right] \right\} = aG(x_{n-1}, x_n, x_n),
\]

so, equation (3.7) will be

\[
G(x_n, x_{n+1}, x_{n+1}) \leq aG(x_{n-1}, x_n, x_n).
\]

So, in each case we have

\[
G(x_n, x_{n+1}, x_{n+1}) \leq qG(x_{n-1}, x_n, x_n)
\]

where \( q = \max \left\{ a, \frac{b}{1 - 2b} \right\} \), since \( 0 \leq a < 1 \) and \( 0 \leq b < \frac{1}{3} \), then \( q < 1 \) and by repeated application of (3.8) we have

\[
G(x_n, x_{n+1}, x_{n+1}) \leq q^n G(x_0, x_1, x_1).
\]

Then, for all \( n, m \in \mathbb{N} \); \( n < m \) we have by repeated use of the rectangle inequality and equation (3.9) that

\[
G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})
\]
+ G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\
\leq (q^n + q^{n+1} + \cdots + q^{m-1})G(x_0, x_1, x_1) \leq \frac{q^n}{1-q} G(x_0, x_1, x_1),

then, \( \lim G(x_n, x_m, x_m) = 0 \), as \( n, m \to \infty \), so \( (x_n) \) is a \( G \)-Cauchy sequence. Due to the completeness of \( (X, G) \), there exists \( u \in X \) such that \( (x_n) \) is \( G \)-convergent to \( u \) in \( (X, G) \).

Suppose that \( T(u) \neq u \), then

\[
G(x_n, T(u), T(u)) \\
\leq \max \big\{ aG(x_{n-1}, u, u), b\big[ G(x_{n-1}, x_n, x_n) + 2G(u, T(u), T(u)) \big], \\
b\big[ G(x_{n-1}, T(u), T(u)) + G(u, T(u), T(u)) + G(u, x_n, x_n) \big] \big\}.
\]

Taking the limit as \( n \to \infty \), and using the fact that the function \( G \) is continuous in its variables, we get

\[
G(u, T(u), T(u)) \leq 2bG(u, T(u), T(u)),
\]

since \( 2b < 1 \), this contradiction implies that \( u = T(u) \).

To prove uniqueness, suppose that \( v \neq u \) such that \( T(v) = v \), then

\[
(3.10) \quad G(u, v) \leq \max \big\{ aG(u, v, v), b\big[ G(u, v, v) + 2G(v, v, v) \big], \\
b\big[ G(u, v, v) + G(v, u, u) \big] \big\}
\]

but, by \( (G5) \), we have \( G(v, u, u) \leq 2G(u, v, v) \), therefore

\[
bG(u, v, v) + bG(v, u, u) \leq 3bG(u, v, v)
\]

then equation \( (3.10) \) becomes \( G(u, v) \leq \max \big\{ aG(u, v, v), b\big[ G(u, v, v) + G(v, v, u) \big] \big\} \leq cG(u, v, v) \), where \( c = \max \{ a, 3b \} \), since \( a < 1 \) and \( b < \frac{1}{3} \), then \( c < 1 \), this implies that \( u = v \).

To show that \( T \) is \( G \)-continuous at \( u \), let \( (y_n) \subseteq X \) be a sequence such that \( \lim (y_n) = u \). Then

\[
(3.11) \quad G(T(y_n), T(u), T(y_n)) \\
\leq \max \big\{ aG(y_n, u, y_n), b\big[ G(u, T(u), T(u)) + 2G(y_n, T(y_n), T(y_n)) \big], \\
\]
But, by (G5), we have
\begin{equation}
G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n)).
\end{equation}
So,
\begin{equation}
2G(y_n, T(y_n), T(y_n)) \leq \left\{ G(y_n, u, u) + G(u, T(y_n), T(y_n)) + G(y_n, T(y_n), T(y_n)) \right\},
\end{equation}
Hence, equations (3.11) will be
\begin{equation}
G(T(y_n), u, T(y_n)) \leq \max \left\{ aG(y_n, u, y_n), b[G(y_n, u, u) + G(u, T(y_n), T(y_n)) + G(y_n, T(y_n), T(y_n))] \right\}.
\end{equation}
Again from equation (3.12) we will get
\begin{align*}
b[G(y_n, u, u) + G(u, T(y_n), T(y_n)) + G(y_n, T(y_n), T(y_n))] & \leq 2bG(y_n, u, u) + 2bG(u, T(y_n), T(y_n)).
\end{align*}
So, equation (3.13) becomes
\begin{equation}
G(T(y_n), u, T(y_n)) \leq \max \left\{ aG(y_n, u, y_n), 2bG(y_n, u, u) + 2bG(u, T(y_n), T(y_n)) \right\}.
\end{equation}
Then, from equation (3.14) we deduce two cases
Case (1): \( G(u, T(y_n), T(y_n)) \leq aG(y_n, u, y_n) \), or
Case (2): \( G(u, T(y_n), T(y_n)) \leq \frac{2b}{1-2b}G(y_n, u, u) \).
In each case, taking the limit as \( n \to \infty \), we see that
\begin{align*}
G(u, T(y_n), T(y_n)) & \to 0
\end{align*}
and so, by Proposition (3), we have \( T(y_n) \to u = Tu \). which implies that \( T \) is \( G \)-continuous at \( u \).
Example 3. Let \( X = [0, 1] \) with the \( G \)-metric defined as follows

\[
G(x, y, z) = \max \{ |x - y|, |y - z|, |x - z| \}, \quad \forall x, y, z \in X.
\]

Define \( T : X \to X \) by \( T(x) = \frac{x}{2}, \forall x \in X \). Then

1. \((X, G)\) is complete \( G \)-metric space.
2. \( T \) satisfies condition (3.1) of Theorem (3.1), for \( a = \frac{3}{4} \) and \( b = \frac{3}{10} \).
3. \( T \) has a unique fixed point \( x = 0 \).

Corollary 1. Let \((X, G)\) be a complete \( G \)-metric space, let \( T : X \to X \) be a mapping which satisfies the following condition for some \( m \in \mathbb{N} \) and for all \( x, y \in X \).

\[
G(T^m(x), T^m(y), T^m(y)) \leq \max \left\{ aG(x, y, z), b\left[ G(x, T^m(x), T^m(x)) + 2G(y, T^m(y), T^m(y)) \right] \right\}
\]

where \( 0 \leq a < 1 \) and \( 0 \leq b < \frac{1}{3} \). Then \( T \) has a unique fixed point, say \( u \), also \( T^m \) is \( G \)-continuous at \( u \).

Proof. From previous theorem, we have \( T^m \) has a unique fixed point (say \( u \)), that is \( T^m(u) = u \). But \( T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u)) \), so \( T(u) \) is another fixed point for \( T^m \) and by uniqueness \( Tu = u \).

Theorem 3.2. Let \((X, G)\) be a complete \( G \)-metric space, let \( T : X \to X \), be a mapping which satisfies the following condition for all \( x, y, z \in X \).

\[
G(T(x), T(y), T(z)) \leq \max \left\{ aG(x, y, z), b\left[ G(x, T(x), T(x)) + G(y, T(y), T(y)) + G(z, T(z), T(z)) \right] \right\}
\]

where \( 0 \leq a < 1 \), and \( 0 \leq b < \frac{1}{3} \). Then \( T \) has a unique fixed point, say \( u \), also \( T \) is \( G \)-continuous at \( u \).

Proof. Taking \( z = y \) in condition (3.16), it becomes condition (3.1), so, the proof follows from Theorem (3.1).
Example 4. Let $X = [0, 1]$ with the $G$-metric defined as follows

$$G(x, y, z) = |x - y| + |y - z| + |x - z|, \quad \forall x, y, z \in X.$$ 

Define $T : X \to X$ by $Tx = \frac{x}{2}, \forall x \in X$. Then

(1) $(X, G)$ is complete $G$-metric space.

(2) $T$ satisfy condition (3.16) of Theorem (3.2), for $a = \frac{1}{3}$ and $b = \frac{32}{100}$.

(3) $T$ has a unique fixed point $x = 0$.

Corollary 2. Let $(X, G)$ be a complete $G$-metric space and $T : X \to X$ be a mapping which satisfies the following condition for all $x, y \in X$

(3.17) 

$$G(T(x), T(y), T(y)) \leq \max \left\{ aG(x, T(x), T(x)) + bG(y, T(y), T(y)) \right\},$$

where $0 \leq a < 1$, and $0 \leq b < \frac{1}{3}$, then $T$ has a unique fixed point, say $u$, also $T$ is $G$-continuous at $u$.

Proof. Every map satisfies condition (3.17), will satisfy condition (3.1), in theorem (3.1), so, the proof follows from theorem (3.1).

Theorem 3.3. Let $(X, G)$ be a complete $G$-metric space, let $T : X \to X$, be a mapping which satisfies the following condition for all $x, y \in X$.

(3.18) 

$$G(T(x), T(y), T(y)) \leq k \max \left\{ [G(x, T(x), T(x)) + 2G(y, T(y), T(y))] ight\},$$

where $k \in [0, 1/4)$, then $T$ has unique fixed point, say $u$, also $T$ is $G$-continuous at $u$.

Proof. Suppose that $T$ satisfies the condition (3.18), then for all $x, y \in X$, we have

(3.19) 

$$G(T(x), T(y), T(y)) \leq k \max \left\{ [G(x, T(x), T(x)) + 2G(y, T(y), T(y))] \right\},$$
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\[ G(x, T(y), T(y)) + G(y, T(y), T(y)) + G(y, T(x), T(x)) \]
\[ G(y, T(x)) + G(y, T(y)) + G(x, x, T(y)) \]

and

\[ G(T(y), T(x), T(x)) \leq k \max \left\{ \left[ G(y, T(y), T(y)) + 2G(x, T(x), T(x)) \right], \right. \]
\[ G(y, T(x), T(x)) + G(x, T(x), T(x)) + G(x, T(y), T(y)) \]
\[ G(x, x, T(y)) + G(x, x, T(x)) + G(y, y, T(x)) \right\} \]

Suppose that \((X, G)\) is symmetric, then by the definition of metric \((X, d_G)\) and equations (2.1), (3.19), we get the following:

\[ d_G(Tx, Ty) = 2G(Tx, Ty) \]
\[ \leq 2k \max \left\{ \left[ \frac{1}{2} d_G(x, Tx) + d_G(y, Ty) \right], \right. \]
\[ \left. \frac{1}{2} \left[ d_G(x, Tx) + d_G(y, Ty) + d_G(y, Tx) \right] \right\} \]

for all \(x, y \in X\).

We see that we have different possible cases and in each one the sum of all coefficients is less 1, also each case will be a special case of condition (2.3) of theorem (2.1), therefore the existence and uniqueness of the fixed point follows from theorem (2.1).

However, if \((X, G)\) is not symmetric then by the definition of the metric \((X, d_G)\), (2.2), (3.19) and (3.20) we get

\[ d_G(Tx, Ty) = G(Tx, Ty) + G(Ty, Tx, Tx) \]
\[ \leq k \max \left\{ \left[ \frac{2}{3} d_G(x, Tx) + \frac{4}{3} d_G(y, Ty) \right], \right. \]
\[ \left. \frac{2}{3} \left[ d_G(x, Ty) + d_G(y, Tx) + d_G(y, Ty) \right] \right\} \]
\[ + k \max \left\{ \left[ \frac{2}{3} d_G(y, Ty) + \frac{4}{3} d_G(x, Tx) \right], \right. \]
\[ \left. \frac{2}{3} \left[ d_G(x, Ty) + d_G(y, Tx) + d_G(x, Tx) \right] \right\} \]
for all \(x, y \in X\).

Again from equation (3.22) we have different possible cases, and in each one the sum of the coefficient is equal \(4k\) which less than 1, also each case will be a special case of condition (2.3) (in theorem (2.1)), therefore the existence and uniqueness of the fixed point follows from Hardy–Rogers theorem (Theorem (2.1)).

To show that \(T\) is \(G\)-continuous at \(u\), let \((y_n) \subseteq X\) be a sequence such that \(\lim (y_n) = u\), then, from (3.18) we deduce that

\[
G\left(T(y_n), T(u), T(y_n)\right) 
\leq k \max \left\{ \begin{array}{c} 2G\left(y_n, T(y_n), T(y_n)\right) + G\left(u, T(u), T(u)\right) \\
G\left(y_n, T(u), T(u)\right) + G\left(u, T(y_n), T(y_n)\right) \\
G\left(y_n, T(y_n), T(y_n)\right) + G\left(y_n, y_n, u\right) \\
G\left(y_n, y_n, T(y_n)\right) + G\left(u, u, T(y_n)\right) \end{array} \right\},
\]

But, from (G5), we have

\[
2G\left(y_n, T(y_n), T(y_n)\right) 
\leq G(y_n, u, u) + G\left(u, T(y_n), T(y_n)\right),
\]

hence, from (3.24), we get

\[
G\left(y_n, T(y_n), T(y_n)\right) 
\leq G(y_n, u, u) + G\left(u, T(y_n), T(y_n)\right) + G\left(y_n, T(y_n), T(y_n)\right).
\]

Then, equation (3.23) becomes

\[
G\left(T(y_n), u, T(y_n)\right) \leq k \max \left\{ \begin{array}{c} G(y_n, u, u) + G\left(u, T(y_n), T(y_n)\right) \\
G\left(y_n, T(y_n), T(y_n)\right) + G\left(y_n, y_n, u\right) \\
G\left(y_n, y_n, T(y_n)\right) + G\left(u, u, T(y_n)\right) \end{array} \right\}.
\]

We see that equation (3.25) implies two cases

Case (1):

\[
G\left(T(y_n), u, T(y_n)\right) 
\leq k \left\{ G(y_n, u, u) + G\left(u, T(y_n), T(y_n)\right) + G\left(y_n, T(y_n), T(y_n)\right) \right\},
\]
again, equation (3.24) implies that

\[ G(\ u, T(y_n), T(y_n)) + G(\ y_n, T(u), T(y_n)) + G(\ y_n, T(y_n), T(y_n)) \]

\[ \leq 2G(y_n, u, u) + 2G(\ u, T(y_n), T(y_n)) \]

then, equation (3.26), becomes

\[ G(\ u, T(y_n), T(y_n)) \leq \frac{2k}{1 - 2k} G(y_n, u, u) \]

Case (2):

\[ G(\ T(y_n), u, T(y_n)) \]

\[ \leq k \{ G(y_n, y_n, u) + G(\ y_n, y_n, T(y_n)) + G(\ u, u, T(y_n)) \} , \]

but, from (G5), we have

\[ G(Ty_n, y_n, y_n) \leq G(Ty_n, u, u) + G(u, y_n, y_n) \]

therefor, equation (3.29), becomes

\[ G(\ T(y_n), u, T(y_n)) \leq k \{ 2G(u, y_n, y_n) + 2G(\ u, u, T(y_n)) \} , \]

also from (G5), we have

\[ G(u, y_n, y_n) \leq 2G(y_n, u, u) \]

and

\[ G(\ u, u, T(y_n)) \leq 2G(\ u, T(y_n), T(y_n)) . \]

then, from equations (3.32) and (3.33) we see that equation (3.31), becomes

\[ G(\ T(y_n), u, T(y_n)) \leq k \{ 4G(y_n, u, u) + 4G(\ u, T(y_n), T(y_n)) \} \]

which implies that,

\[ G(\ T(y_n), u, T(y_n)) \leq \frac{4k}{1 - 4k} G(y_n, u, u) . \]

In equations, (3.28) and (3.35), taking the limit as \( n \to \infty \), we see that

\[ G(\ u, T(y_n), T(y_n)) \to 0 \]

and so, by Proposition 3, we have \( T(y_n) \to u =Tu \).

which implies that \( T \) is \( G \)-continuous at \( u \). \( \square \)
Corollary 3. Let \((X,G)\) be a complete \(G\)-metric space, and \(T : X \to X\) be a mapping which satisfies the following condition for some \(m \in \mathbb{N}\) and for all \(x,y \in X\).

\[
G(T^m(x), T^m(y), T^m(y)) \\
\leq k \max \left\{ \begin{array}{l}
[G(x, T^m(x), T^m(x)) + 2G(y, T^m(y), T^m(y))] \\
[G(x, T^m(y), T^m(y)) + G(y, T^m(y), T^m(y)) + G(y, y, T^m(x))] \\
+ G(y, T^m(x), T^m(x)), \left[ G(y, y, T^m(x)) \\
+ G(y, y, T^m(y)) + G(x, x, T^m(y)) \right] \end{array} \right\}
\]

where \(k \in [0, 1/4)\), then \(T\) has unique fixed point, say \(u\), also \(T^m\) is \(G\)-continuous at \(u\).

Proof. The proof follows from previous theorem and the same argument as in Corollary (1). \(\square\)

Theorem 3.4. Let \((X,G)\) be a complete \(G\)-metric space, let \(T : X \to X\), be a mapping which satisfies the following condition, for all \(x,y,z \in X\).

\[
G(T(x), T(y), T(z)) \\
\leq k \max \left\{ \begin{array}{l}
[G(x, T(x), T(x)) + G(y, T(y), T(y)) + G(z, Tz, Tz)] \\
[G(x, T(y), T(y)) + G(y, T(z), T(z)) + G(z, T(x), T(x)), \left[ G(y, y, T(x)) \\
+ G(z, z, T(y)) + G(x, x, T(z)) \right] \end{array} \right\}
\]

where \(k \in [0, 1/4)\), then \(T\) has unique fixed point, say \(u\), and \(T\) is \(G\)-continuous at \(u\).

Proof. Taking \(z = y\) in condition (3.37) it becomes condition (3.18), so, the proof follows from Theorem (3.3). \(\square\)

Corollary 4. Let \((X,G)\) be a complete \(G\)-metric space, and let \(T : X \to X\) be a mapping which satisfies the following condition, for all \(x,y \in X\).

\[
G(T(x), T(y), T(y)) \leq k \max \left\{ \begin{array}{l}
[G(x, T(x), T(x)) + G(y, T(y), T(y))] \\
[G(x, T(y), T(y)) + G(y, T(x), T(x)), \left[ G(y, y, T(x)) + G(x, x, T(y)) \right] \end{array} \right\}
\]

where \(k \in [0, 1/4)\), then \(T\) has a unique fixed point, say \(u\), and \(T\) is \(G\)-continuous at \(u\).
Proof. Every map satisfies condition (3.38), will satisfy condition (3.18), in theorem (3.3).

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