Time dependent analysis of short-range monopole wake fields driven by accelerated electron beams

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Abstract

Combinations of integral transforms and Green’s function are employed in order to obtain the distribution of the short-range monopole wake fields driven by accelerated electron beams in a cylindrical cavity. This constituted an alternative but simpler and more versatile method to the normal mode expansion which we used earlier to solve for such a distribution (Salah and Dolique, Nucl. Instr. and Meth. A 431 (1999) 27). Whereas direct comparisons of the resulting analytical expressions are found difficult to directly compare, numerical illustration of results obtained from both approaches show excellent agreements and hence confirm the fact that both approaches are equivalent. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In a recent paper [1], we utilized the time dependent normal mode analysis procedure in order to study the wake field distribution of strongly accelerated beams in cylindrical cavities. This is a different field from the ultra relativistic one studied by the same normal mode analysis previously by [2–10]. Specifically, strongly accelerated wake fields are produced by combination of the electromagnetic reaction of the acceleration structure walls with the beam generated field. As was pointed out in Ref. [1], the most important difference between both fields is the variation of the beam velocity with time in the accelerated field.

In this paper, we describe analytically, the wake field driven by a strongly accelerated beam. As was justified in Ref. [1], we limit our modelling to a “pill-box” type cavity. For an application, we shall adopt the photoinjector of the ELSA facility (CEA, Bruyeres le Châtel), as shown in Fig. 1 [1,11,12].

The analytical expression of the wake field map \((E, B)(x, t)\) is obtained by using both integral transform and Green’s function methods, Maxwell’s equations are solved in the space-time domain, taking into account the boundary conditions on the cavity walls, without recourse to developments in normal modes.

2. Dynamics of electron beam

The velocity and the acceleration of electrons are obtained by using the fundamental relation of
dynamics under relativistic form [1]

\[ \sum \ddot{\mathbf{F}} = \frac{d\vec{p}}{dt} \]  

(1)

where \( \vec{p} = \Gamma m \vec{v} \), \( \Gamma \) is the relativistic mass factor. The velocity \( \vec{v}(z, t) \) and the acceleration \( (z, t) \) are shown be parallel to \( \vec{E}_0 \) and independent of time [1,11]

\[ \vec{v}(z, t) = \beta(z)\vec{u}_z \]

\[ \gamma(z) = \frac{1 + Hz}{1 + Hz(t)} \]  

\[ z(t) = \frac{1}{H} \left( \sqrt{1 + (Hc(t - t_z))^2} - 1 \right) \]  

\[ H^{-1} = \frac{mc^2}{eE_0} \]  

where \( e \) and \( m \) are the charge and the mass of the electron respectively, \( c \) is the velocity of light, \( z(t) \) is the longitudinal coordinates of electron at time \( t \), \( t_z \) time at which the element \( z \) of the beam leaves the photo cathode.

3. Brief outline of calculations

3.1. Starting equations modelling

We solve Maxwell’s equations for potentials, in Coulomb gauge

\[ \nabla^2 \Phi(r, z, t) - \frac{\rho}{\varepsilon_0} = 0 \]  

(7)

\[ \Box \vec{A}(r, z, t) = \mu_0 \vec{j} - \frac{1}{c^2} \frac{\partial}{\partial t}(\nabla \Phi) \]  

(8)

in the axisymmetric cylindrical cavity D (“Pill-box”): \( 0 \leq z \leq \delta; 0 \leq r \leq R \), where \( \Phi \) and \( \vec{A} \) are the scalar and vector potentials, respectively, \( \Box \) is the Alembertian operator, \( \rho(r, z, t) \) and \( \vec{j}(r, z, t) \) are the current and charge density given, respectively [1],

\[ \rho(r, z, t) = \frac{I\sigma(z, t)}{\pi a^2 \beta(z)c}[1 - H(r - a)] \]

\[ \vec{j}(r, z, t) = \beta(z)c\rho(r, z, t)\vec{u}_z. \]  

(9)

This system of equations describe a radialy uniform beam of radius \( a \) and velocity \( v(z) = \beta(z)c \), carrying a total current \( I \), is assumed to be radially uniform. Its axial current profile \( \sigma(z, t) \) is uniform, with stiff front and back, and time length \( \tau \).

These equations are supplemented with appropriate boundary conditions. For the initial conditions at the beginning of the emission, \( t = 0 \), we assume \( \Phi(r, z, t) = 0 \), \( \vec{A}(r, z, t) = 0 \); \( \partial \Phi/\partial t = 0 \) and \( \partial \vec{A}/\partial t = 0 \). At the location \( z = g \), where the anode is located, we have to satisfy: Dirichlet conditions for \( \Phi \) and the tangential component of \( \vec{A} \): \( \Phi(r, z = g, t) = 0 \), \( \vec{A}_t(r, z = g, t) = 0 \), and Neuman conditions for the normal component of \( \vec{A} \): \( \partial \vec{A}_n \times (r, z = g, t) / \partial n = 0 \).

3.2. Analytical results

Eqs. (7) and (8) are now solved via the methods of integral transform and Green’s function techniques in the Coulomb-gauge. According to symmetries, the only non-vanishing components, in cylindrical coordinates, are \( E_z \), \( E_r \) and \( B_\theta \).
After lengthy algebraic reductions and manipulations we arrive at:

### 3.2. Longitudinal E-field

\[ E_{\parallel}(R, Z, T) = \frac{2I}{\pi \varepsilon_0 c A^2} \sum_{p=1}^{\infty} \frac{\cos(kpz)}{kp} \]

\[ \times \left\{ \int_{\max(0, T-\tau)}^{\infty} \sin[kp(\sqrt{1 + u^2} - 1)] \, du \right\} \]

\[ \times \left\{ 1 - (Ak p) \sum_{p=1}^{\infty} \frac{\cos(kpz)}{kp} \sin[kp(\sqrt{1 + u^2} - 1)] \right\} \]

\[ \times \left\{ \begin{array}{l}
K_1(Ak p)I_1(Rk p), \quad 0 \leq R \leq A \leq \Re' \\
I_1(Ak p)K_1(Rk p), \quad 0 \leq A \leq R \leq \Re' \end{array} \right. \]

(10)

### 3.2.2. Transverse E-field and magnetic field \( B_0 \)

\[ E_{\perp}(R, Z, T, \tau) = -\frac{2I}{\pi \varepsilon_0 c A g} \int_{0}^{\infty} x J_0(Rx) \sum_{p=1}^{\infty} \frac{\sin(kpz)}{(kp)^2} m_{Ak p} \left( \frac{x}{kp} \right) \]

\[ \times \left\{ \int_{0}^{T} \left[ \cos[x(T - T')] \left( \frac{\sqrt{1 + T'^2}}{2} \right) \right. \right. \]

\[ \left. - \sqrt{1 + (T' - \tau)^2} \right] dT' + \sum_{p=1}^{\infty} \frac{\cos(kpz)}{kp} \sqrt{x^2 + (kp)^2} \]

\[ \times \left\{ \begin{array}{l}
J_0(Rx) m_{Ak p} \left( \frac{x}{kp} \right) + \frac{2x^2}{x^2 + (kp)^2} J_1(Ax) J_1(Rx) \\
\sin[kp(\sqrt{1 + T'^2} - 1)] \right. \]

\[ \left. - \sin[kp(\sqrt{1 + (T' - \tau)^2} - 1)] \right\} dT' \right\} \right. \}

\[ dx \]

\[ \times \int_{0}^{T} \cos \left[ \sqrt{x^2 + (kp)^2}(T - T') \right] \]

\[ \times \left[ \sin[kp(\sqrt{1 + T'^2} - 1)] \right. \]

\[ - \sin[kp(\sqrt{1 + (T' - \tau)^2} - 1)] \left. \right\} dT' \right\} \]

(11)

where \( I_0, I_1, K_0 \) and \( K_1 \) are modified Bessel functions, \( k = \pi / G, Z = Hz, A = Ha, G = Hg, R = Hr, \Re' = H9R, u = \sqrt{Z(Z+2)}, \tau' = Hc \tau \) and \( I \) is the total current carrying by the beam.

### 4. Application to the “ELSA” photo injector

As an example for our present applications, we chose the following parameters which are also consistent with the “ELSA” facility: \( I = 100 \text{ Am}, \pi a^2 = 1 \text{ cm}^2 (a \text{ is the radius of the beam }) \) \( E_0 = 30 \text{ and } 10 \text{ MV/m}, \ r_0 = 2 \text{ cm} (r_0 \text{ is the exit aperture radius}), g = 6 \text{ cm and } \tau = 30 \text{ ps}. \)
The field maps obtained by the method described in this paper are compared to those that we recently published for the photo injector "ELSA" where the analytical expression of the \((E, B) (x, t)\) map have been obtained by using the time dependent normal modes analysis [1]. This test has been successfully carried out, as shown in Figs. 2–3.

Fig. 2 shows, for \(E_0 = 30\) MV/m and \(t = \tau\), i.e. at the end of the photo emission, the beam-generated axial \(E\)-field \(E_x\), on the axis \((r = 0)\) as a function of the reduced abscissa \(Z = Hz\). At \(t = \tau\), the anode is not yet reached by an electromagnetic signal, so that the cathode is the only wall which plays a role in the wake field [1].

Fig. 3 shows \(E_x(r = a, z, t)\) for \(E_0 = 30\) MV/m and \(t = t_g = 250\) ps, where \(t_g\) is the time at which the beam head reaches the photo injector exit \(z = g\).

Fig. 4 shows \(B_y\) in front of the beam for \(E_0 = 10\) MV/m and \(t = \tau\), causality prevents any beam's field influence at a distance from the emissive cathode greater than \(ct\).

Fig. 5 shows \(E_x(r, z, t)\) for \(t = t_g/2 = 164\) ps and \(E_0 = 10\) MV/m, behind, inside and in front of the beam. Causality prevents any beam's field influence at a distance greater than \(ct\).

5. Conclusion

In an RF gun, photo injectors used as high quality, high intensity and short beam source, one is faced with a new wake field problem: the wake field of strongly accelerated particles which, extracted from the cathode with thermal velocities, become relativistic before they leave the cavity.
For a cylindrical cavity, the wake field map corresponding to this situation is analytically deduced from Maxwell’s equations by both integral transform and Green’s function techniques. This constituted an alternative but simpler and more versatile method to the normal mode expansion, which we used earlier to solve for such a distribution [1]. Whereas direct comparisons of the resulting analytical expressions are found difficult to directly compare, numerical illustration of results obtained from both approaches show excellent agreements and hence confirm the fact that both approaches are equivalent.

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References