Analytical and numerical investigations of the evolution of wake fields of accelerated electron beams encountering cavity discontinuities in laser-driven RF-free electron laser photoinjector

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Received 19 October 2003; received in revised form 14 April 2004; accepted 20 May 2004
Available online 14 July 2004

Abstract

We report a rigorous derivation of an analytical expression for the wake fields of accelerated electron beams that takes into account the discontinuity of the accelerating cavities of linear accelerators. The approach employs Fourier’s transforms, Green’s function techniques, and Laurent series in order to solve the full set of Maxwell’s equations in the Lorentz gauge. This represents a significant step toward a realistic description of the evolution of wake fields in linear accelerators. The expression is applied to the “ELSA” photo injector to obtain relevant numerical results.
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PACS: 41.75.—i; 41.75 Lx; 41.60.—m

Keywords: Accelerated electron beam; Wake field; RF-photoinjector; Cavity discontinuity

1. Introduction

A bunch of charged particles traversing a resonant cavity interacts with the surrounding structure and in the process induces electromagnetic fields commonly called wake fields. These fields act back on the bunch influencing its dynamics and significantly deteriorating its quality, which is usually measured by the emittance growth and energy dispersion. Knowledge of these fields, therefore, is necessary for the calculation of the coupling impedance [1] as well as the evaluation of the energy loss of the bunched beam [2].

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0168-9002/ - see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.nima.2004.05.129
Different approaches to measure experimentally or to calculate numerically the wake fields driven by an electron beam passing an infinitely long periodic sequence of radio-frequency (RF) accelerating cavities have been reported in the literature [3–12]. These traditional approaches assume that the beam has a constant velocity equal to the speed of light. This assumption may be regarded as a rather reasonable first approximation when the beam pulse is sufficiently far from the cathode. It cannot be retained for the beam pulse which had just been emitted. In an RF-gun, the electrons at the beam head, about 1 cm from the cathode, are already accelerated by the RF-field to relativistic velocity while at the beam back, next to the cathode, one may find electrons with thermal velocities. Therefore, the beam velocity is far from being constant. Moreover, these approaches do not discuss the effect of the accelerating cavities discontinuities on the wake fields, and are, therefore, unrealistic. In these conditions, the electromagnetic field map inside the beam pulse strongly differs from the map which would be deduced from an electrostatic calculation made in the beam centroid frame.

In fact, the main difficulty in the numerical time domain approach is to formulate proper boundary conditions at the open end of the photoinjector, which takes the variation of the velocity into account. Recently, we have published theoretical modelling that takes into account the effect of the acceleration phase in the gun [13,14], and that presents valuable analytical descriptions of the wake field driven by a strongly accelerated electron beam in an RF-gun using two different approaches. The first employed a modified time-dependent normal mode analysis [13], while the second employed a time-dependent solution of Maxwell’s equations in the Coulomb gauge using both integral transforms and Green’s function techniques [14]. Both approaches assumed that the perturbation of the field map by the exit hole is neglected as long as \( r_0/R \ll 1/3 \) [15,16], where \( r_0 \) and \( R \) are the hole and cavity radii, respectively (Fig. 1). The phase where the beam penetrates the exit aperture is the focus of this paper.

A first consequence of causality principle—which prevents any beam’s field influence at a distance from the emissive cathode greater than \( ct \)—is thus to restrict the part of photoinjector walls able to contribute to beam wake [17]. For an RF-field amplitude \( E_0 \) of some tens of MV/m, (say 30 MV/m) and beam time length...
(say 30 ps), it takes 350 ps for the beam head to cross a photoinjector cavity having a gap $g$ of say 6 cm. Within this time, the radial wall ($r = R = 56$ cm) is not reached by any electromagnetic signal coming from the beam. As for the transverse walls, only a fraction of them: $z = 0, z = g, r < r_{\text{max}}$ can generate a wake able to influence the beam, whatever the beam pulse duration $\tau$ and $E_0$. Therefore a cylindrical pillbox style cavity modelling: $(0, g)(0, r_{\text{max}})$ is well found with any $r_{\text{max}} > c t_g$, where $t_g$ is the time at which the beam head reaches the photoinjector exit.

For the sake of simplicity, changes in the shape or size of the beam due to the self-field force, the wake-field force, and the jump in the cross-section of the photoinjector will be neglected. It is an idealised approach that may be useful as a first step.

2. Theoretical framework

The electromagnetic field will be obtained by adding the synchronous field and a radiation field with undetermined coefficients in the cavity and the drift tube, respectively. These coefficients will be determined by writing the appropriate boundary conditions: (a) on the photoinjector anode ($z = g, r_0 \leq r \leq r_0'$) for the coefficients relative to the field inside the photoinjector, where $r_0'$ is the part of the anode wall around the exit hall reached by the exciting signal of the beam, (b) on the aperture ($z = g, 0 \leq r \leq r_0$) for the radiation coefficients inside the tube, (c) the continuity of the ($E, B$)($X, t$) map and their derivatives at $z = g$.

2.1. Wake field inside the cavity

2.1.1. Synchronous field

This is the particular solution of Maxwell’s equations in Lorentz gauge:

$$
\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0
$$

(1)

$$
\square \vec{A}(r, z, t) = \mu_0 \vec{j}
$$

(2)

$$
\square \Phi(r, z, t) = \frac{\rho}{\epsilon_0}
$$

(3)

where $\square$ is the Alembertian operator, $\Phi$ and $\vec{A}$ are the scalar and vector potentials, respectively, $\epsilon_0$ and $\mu_0$ are the permittivity and permeability of free space, respectively, $c$ is the speed of light, and $\rho(r, z, t)$ and $\vec{j}(r, z, t)$ are the current and charge density, respectively, and are given by [13]

$$
\rho(r, z, t) = \frac{I \sigma(z, t)}{\pi a^2} \beta(z) c [1 - \Theta(r - a)]
$$

$$
\vec{j}(r, z, t) = \beta(z) c \rho(r, z, t) \vec{u}_z
$$

(4)

where $\Theta$ is the Heavside step function.

They describe a radially uniform beam of radius $a$ and velocity $v(z) = \beta(z) c$, carrying a total current $I$. Its axial current profile $\sigma(z, t)$ is uniform, with stiff front and back, and time length $\tau$. However, the velocity $\beta(z)$ is shown to be parallel to the accelerated field $E_0$ and independent of time [13]:

$$
\beta(z) = \sqrt{1 + H(z(t))^2} - 1
$$

$$
\frac{1}{1 + H(z(t))}
$$

(5)
with

\[ z(t) = \frac{1}{H} \left( \sqrt{1 + [Hc(t - t_z)]^2} - 1 \right) \]  

(6)

and

\[ H^{-1} = \frac{mc^2}{eE_0} \]  

(7)

Here, \( e \) and \( m \) are the charge and the mass of the electron, respectively; \( z(t) \) is the longitudinal coordinate of the electron at time \( t \), and \( t_z \) is the time at which the element of the beam at location \( z \) leaves the photocathode.

For points on the conducting surface \((r_0 \leq r \leq \infty, z = g)\) the potential is zero because this surface is grounded. For these points, we have to assign Dirichlet conditions for \( \Phi \) and the tangential components of \( \vec{A} \) at the location \( z = g \) where the anode is located, therefore, we have to satisfy

\[ \Phi(r_0 \leq r \leq \infty, z = g) = 0 \]  

(8)

\[ A_r(r_0 \leq r \leq \infty, z = g) = 0 \]  

(9)

and Neumann condition for the normal component of \( \vec{A} \)

\[ \frac{\partial A_z}{\partial z}(r_0 \leq r \leq \infty, z = g) = 0 \]  

(10)

Owing to causality, the radial wall at the location \( r = R \) will not be considered because they are not reached by an electromagnetic signal coming from the beam. The influence of the transverse wall at the location \( z = 0 \) (where the cathode is located) is lessened considerably as the beam moves away from the cathode. According to the azimuthal symmetry of the geometry, the only non-vanishing components, in cylindrical coordinates, are \( E_z \), \( E_r \), and \( B_\theta \). For \( B_\theta \) the condition \((1/r \partial/\partial r(\rho B_\theta))_{r=R} = 0\) must be satisfied which impose \( A_r(\infty) = 0 \).

Under these conditions, the solution of Eq. (2) can be written as a development of Dini series of order zero.

\[ A_z(r, z, t)_{\text{syn}} = \sum_{n=1}^{\infty} a_n(z, t)_{\text{syn}} J_0 \left( j_n \frac{r}{R} \right) \]  

\[ a_n(z, t)_{\text{syn}} = \frac{2}{J_1(j_n)} \int_0^\infty A_z(r, z, t)_{\text{syn}} r J_0 \left( j_n \frac{r}{R} \right) \, dr \]  

(11)

Substituting Eq. (11) into Eq. (2), and using Fourier’s transforms and Green’s function techniques, one finds for the vector potential

\[ A_z(r, z, t)_{\text{syn}} = \frac{4\mu_0 I_c}{\pi^2 a^2} \sum_{n=1}^{\infty} \frac{J_0 \left( j_n \frac{r}{R} \right) J_1 \left( j_n \frac{r_z(t-g)}{R} \right)}{J_0^2(j_n)} \int_0^\infty \frac{\cos(hz)}{h\omega_n} \left\{ \frac{\sin(hz)}{\omega_n} (1 - \cos(\omega_n(t - t_g))) \right. \]

\[ - \left. \int_{t_z}^t \sin[\omega_n(t - t')] \psi(t' - \tau) \, dt' + \int_0^{t_z} \sin[\omega_n(t - t')] \psi(t') \, dt' \right\} \, dh \]  

(12)

where \( I \) is the total current carried by the beam, \( a \) is the beam radius, \( j_n \) is the \( n \)th zero of the Bessel function \( J_0 \), \( \omega_n = c \sqrt{h^2 + (j_n/R)^2} \), and \( \psi(t) = \sin(hz_f(t)) \), where \( z_f(t) \) is the coordinate of the head of the beam.
By the gauge of Lorentz we get the scalar potential

$$\Phi(r, z, t)_{\text{syn}} = \frac{4I^2}{\pi^2\varepsilon_0 c^2} \sum_{n=1}^{\infty} \frac{J_0(j_n \frac{r}{\sqrt{h}})J_1(j_n \frac{r}{\sqrt{h}})}{j_n J_2^2(j_n)} \int_0^\infty \frac{\sin(hz)}{\omega_n^2} \times \left\{ \left( t - t_0 - \frac{1}{\omega_n} \sin(\omega_n(t - t_0))\psi(t_0) + \int_0^{t_0} [1 - \cos(\omega_n(t - t'))] \psi(t') dt' \right. \right. \left. \left. - \int_0^{t_0} [1 - \cos(\omega_n(t - t'))] \psi(t') dt' + \int_0^{t_0} [1 - \cos(\omega_n(t_0 - t'))] \psi(t') dt' \right\} dh. \right\}$$  

(13)

2.1.2. Radiation field

This is the general solution of Eqs (2) and (3) without source terms $\rho(r, z, t)$ and $j(r, z, t)$. Taking the boundary condition into account, this general solution can be written as

$$A_z(r, z, t)_{\text{rad}} = \mu_0 I^3 \sum_{n=1}^{\infty} \frac{J_0(j_n \frac{r}{\sqrt{h}})}{\sqrt{h}} \int_0^\infty \cos(hz) [C_{A_n}(h) \cos(\omega_n t) + C'_{A_n}(h) \sin(\omega_n t)] dh$$  

(14)

$$\Phi(r, z, t)_{\text{rad}} = \frac{I}{\varepsilon_0 c} \sum_{n=1}^{\infty} \frac{J_0(j_n \frac{r}{\sqrt{h}})}{\sqrt{h}} \int_0^\infty \sin(hz) [C_{\Phi_n}(h) \cos(\omega_n t) + C'_{\Phi_n}(h) \sin(\omega_n t)] dh$$  

(15)

where $C_{A_n}$, $C'_{A_n}$, $C_{\Phi_n}$ and $C'_{\Phi_n}$ are arbitrary dimensionless functions.

By Lorentz gauge we get

$$C_{A_n}(h) = \frac{\omega_n}{\hbar c^2} C'_{\Phi_n}(h)$$  

(16)

$$C'_{A_n}(h) = \frac{\omega_n}{\hbar c^2} C'_{\Phi_n}(h).$$  

(17)

The functions $C_{A_n}$ and $C'_{A_n}$ are expanded in Laurent series

$$C_{A_n}(h) = \sum_{m=1}^{\infty} C_{A_n, m}(h \sqrt{h})^{-2m}$$  

(18)

$$C'_{A_n}(h) = \sum_{m=1}^{\infty} C'_{A_n, m}(h \sqrt{h})^{-2m}.$$  

(19)

$C_{A_n, m}$ and $C'_{A_n, m}$ can be calculated by applying the boundary condition given by Eq. (10) written for $A_z = A_{\text{sync}} + A_{\text{rad}}$, multiplied by $r$ and integrated on $r$ in its validity domain $r_0 \leq r \leq 3r$, one is led to an infinite system of linear algebraic equations

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{mn}(t) C_{A_{mn}} + \beta_{mn}(t) C'_{A_{mn}} = L(t)$$  

(20)

where

$$\alpha_{mn}(t) = \int_0^{3r} \left[ J_1(j_n) - \frac{r_0}{3r} J_1(j_n \frac{r_0}{3r}) \right] \int_0^\infty \frac{\sin(hg) \cos(\omega_n t)}{(h \sqrt{h})^{2m-1}} dh$$  

(21)
2.2.1. Synchronous field

This leads to adopting the following expressions for the vector and scalar potential, respectively

\[ \beta_{nm}(t) = \frac{2}{J_n} \int_0^\infty \frac{\sin(ny)\sin(\omega_n t)}{h_n} \, dh \]

\[ L(t) = \frac{4I_xI_y}{\pi^2 \epsilon_0 c} \sum_{n=1}^\infty \frac{J_1(j_n \frac{r_0}{R})}{j_n^2 J_1(j_n) / j_n^2 J_1(j_n)} \left[ J_1(j_n) - \frac{r_0}{R} J_1(j_n) \right] \]

\[ \times \left\{ \int_0^\infty \frac{\sin(ny)}{\omega_n} \left( \frac{\sin(ny)}{\omega_n} (\cos(\omega_n (t - t_0)) - 1) + \int_0^t \sin(\omega_n (t - t')) \psi(t' - t) \, dt' \right) - \int_0^\infty \sin(\omega_n (t - t')) \psi(t') \, dt' \right\} dh \].

A good convergent is reached by truncating at \( n_{max} = N, m_{max} = M \), with \( N \sim 100 \) and \( M \leq 10 \). The \( 2N(M + 1) \) unknowns are then calculated numerically by solving the system of \( 2N(M + 1) \) equations

\[ \sum_{n=0}^{N} \sum_{m=0}^{M} \beta_{nm}(t) \hat{A}_{nm} + \beta_{nm}(t) \hat{C}_{nm} = L(t) \]

where \( t_i \in [t_0, t_g + \tau] \) and \( i \in [1, 2, \ldots, NM] \).

\( \hat{A}_{nm}, \hat{C}_{nm}, \hat{C}_{nm} \) and \( \hat{C}_{nm} \) being calculated, \( \Phi(r, z, t) \) and \( A(r, z, t) \) are deduced, from which the fields result.

2.2. Wake field inside the drift tube

2.2.1. Synchronous field

The used techniques are the same as in Section 2.1.1. The boundary conditions to be satisfied are now

\[ \Phi(r = r_0, z) = 0 \]

\[ \frac{\partial A_z}{\partial z}(r = r_0, z) = 0. \]

This leads to adopting the following expressions for the vector and scalar potential, respectively

\[ A_z(r, z, t)_{syn} = \frac{\mu_0 I_0}{\pi \epsilon_0 c} \sum_{n=1}^\infty \frac{J_0(j_n \frac{r}{R}) J_1(j_n \frac{r_0}{R})}{j_n^2 J_1(j_n)} \left\{ 1 - \exp\left( -\frac{j_n}{r_0} ((z - g) - \beta c (t - t_0)) \right) \right\} \]

\[ \Phi(r, z, t)_{syn} = \frac{I_0}{\pi \epsilon_0 c} \sum_{n=1}^\infty \frac{J_0(j_n \frac{r}{R}) J_1(j_n \frac{r_0}{R})}{j_n^2 J_1(j_n)} \]

\[ \times \left\{ 2 - \exp\left( -\frac{j_n}{r_0} (z - g) \right) - \exp\left( -\frac{j_n}{r_0} (z - g) - \beta c (t - t_0) \right) \right\} \]

where \( \gamma = \frac{1}{\sqrt{1 - \beta^2 c^2}}. \)

2.2.2. Radiation field

The techniques used are the same as in Section 2.1.2. The boundary conditions to be satisfied are

\( \Phi(r = r_0, z, t_g \leq t \leq t_g + \tau) = 0, A_z(r = r_0, z, t_g \leq t \leq t_g + \tau) = 0 \) and a radiation condition for \( z \to +\infty \). One
finds for the vector and scalar potentials, respectively

\[ A_z(r, z, t)_{\text{rad}} = \frac{\mu_0 I_0}{c} \sum_{n=1}^{\infty} J_0 \left( \frac{j_n}{r_0} \right) \int_0^\infty D_{A_n}(\omega) \exp \left( i \left( \omega t - \sqrt{\frac{\omega^2}{c^2} - \left( \frac{j_n}{r_0} \right)^2} z \right) \right) d\omega \]  

(29)

\[ \Phi(r, z, t)_{\text{rad}} = \frac{I_0}{\pi e_0 c^2} \sum_{n=1}^{\infty} J_0 \left( \frac{j_n}{r_0} \right) \int_0^\infty D_{\Phi_n}(\omega) \exp \left( i \left( \omega t - \sqrt{\frac{\omega^2}{c^2} - \left( \frac{j_n}{r_0} \right)^2} z \right) \right) d\omega \]  

(30)

with \( \omega > 0 \) (wave propagating in the positive \( z \) direction) and \( i = \sqrt{-1} \). By the gauge of Lorentz (Eq. (1)) we get

\[ D_{A_n} = -\frac{1}{c} \frac{\omega}{\sqrt{\left( \omega/c \right)^2 - 1}} D_{\Phi_n}. \]  

(31)

The unknown functions \( D_{\Phi_n} \), expanded in Laurent series, are calculated by writing the field continuity on the aperture \( (z = g, 0 \leq r \leq r_0) \) and integrating on \( r \), in the interval \( 0 \leq r \leq r_0 \).

\( D_{\Phi_n} \) and \( D_{A_n} \) being calculated, \( \Phi(r, z, t) \) and \( A(r, z, t) \) are deduced, from which the fields in the drift tube result.

### 3. Numerical results

To numerically apply our analytical results, we use the parameters of the “ELSA” facility: \( I = 100 \) Am, \( \pi a^2 = 1 \) cm\(^2 \) where \( a \) is the radius of the beam, \( E_0 = 30 \) MV/m, \( r_0 = 2 \) cm, where \( r_0 \) is the exit aperture radius, \( g = 6 \) cm, and \( \tau = 30 \) ps.

The electric and magnetic fields can be obtained from the rebuilt potentials. Figs. 2 and 3 show the beam-generated fields \( E_z \) on the axis \((r = 0)\) and \( E_r \) on the beam surface \((r = a)\), respectively, as function of the reduced abscissa \( Z = Hz \), at \( t = t_g + \tau/4 \). Here \( t_g \) is the time at which the beam head reaches the photo injector exit \( z = g \). \( Z \) is the reduced coordinates, and \( H \) is the characteristic length defined by \( H^{-1} = mc^2/eE_0 \).

![Fig. 2. The beam-generated axial E-field \( E_z \), on the axis \((r = 0)\) as a function of the reduced abscissa \( Z = Hz \) at \( t = t_g + \tau/4 \).](image)
Field discontinuities are observed in both figures at the transition between the cavity and the drift tube. These discontinuities occur at the same time and approximately the same place within the bunch; they might be due to perturbation of the volume charge density distribution within the bunch. In Fig. 2, the discontinuity appears to be caused by the electrons situated at the head of the beam that have entered the drift tube. While the head and rear electrons of the beam bunch interact with a decelerating field, the electrons situated at the middle of the bunch interact with an accelerating field. These accelerating/decelerating fields could split the center of the beam bunch from the bunch. Fig. 3 illustrates the radial field $E_r$ for the electrons situated at the beam surface. This field causes a radial motion that may lead to beam expansion.

Fig. 4 shows the magnetic field $B_\theta$ inside the beam $(r = 3a/5)$ at $t = t_\theta + \tau/2$. The presence of this wake magnetic field causes an azimuthal motion, besides the longitudinal and the radial ones discussed earlier.

Since no experimental data are available to compare our results with, we have tested our results against the theoretical work of Dolique and Coacolo [18] who investigated the self-field $(E, B)$ of the accelerated beam. In their work, the evolution of the self-field is calculated as a direct relativistic electron–electron interaction field obeying the Liénard–Wiechart formulae. Their $E_r$ and $B_\theta$ are also shown in Figs. 3 and 4.
respectively. Clearly, Figs. 3 and 4 show that the beam self-field and the wake field are not identical. This has also been pointed out by Salah and Dolique [13] where they have shown that it is the sum of both the cavity reaction field and the beam self-field that constitute the so called wake field.

Fig. 5 shows $E_z$ again but at $t = t_0 + 3\tau/4$. Now, the electron situated at the beam head interact with an accelerating field, while those situated at the beam back interact with an decelerating field. Fig. 6 repeats Fig. 4 at $t = t_0 + \tau$, where the beam penetrates completely in the drift tube.

**4. Conclusion**

We have investigated both analytically and numerically the wake fields induced by an intense accelerated electron beam while exiting the photoinjector cavity and penetrating the drift tube of a laser-driven RF-electron gun. These wake fields lead to radial and azimuthal motions for the electrons situated in the beam, thus leading to deviations from pure longitudinal motion. These deviations could significantly deteriorate the beam quality by causing energy dispersion and thus limiting the free electron laser efficiency. To avoid
the latter, the beam must be focused by a set of magnetic lenses (e.g. solenoids, quadruples, etc…) that could be properly placed with respect to each other or with respect to the cathode at the exit of the RF-gun. A natural extension to this work would be to consider a full or half-wavelength cell, of which most high-brightness photoinjectors have at least one, and will be the subject of our future work.

Acknowledgements

The author is deeply indebted to Prof. J.-M. Dolique from CEA-PTN, Bruyères-le-Châtel, France, for fruitful discussions of these results.

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