Analysis of the transverse kick to beams in low-frequency photoinjectors due to wakefield effects

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\textbf{A B S T R A C T}

A time domain analysis of the normal modes in a cavity is used to obtain an analytical expression for the transverse momentum imparted to particles within an accelerated electron beam in a low frequency photoinjector. These analytical expressions form the basis of detailed simulations on the transverse momentum imparted to an accelerated beam. This analysis of the wakefields employs a modified form of the Panofsky–Wenzel theorem in which additional velocity dependent effects are taken into account. Simulations are presented for parameters of the ELSA photocathode.

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1. Introduction

The progress of a charged particle beam through a cavity excites an electromagnetic field (e.m) which interacts back on the beam itself. This e.m field can be decomposed into a longitudinal component which influences the energy spread of the beam and a transverse component which causes an emittance dilution of the beam and can give rise to a beam break up instability \cite{1–7}. This e.m field is conveniently represented as a wakefield which results from the influence of the e.m field scattered off the cavity wall and this we refer to as the geometric wakefield, or from Ohmic losses in the walls. The latter gives rise to resistive wall instability \cite{8,9} and will not be studied in this paper. The focus of this paper is on the geometric wakefield. In either case, the wakefield may be viewed as an image charge setup by the beam which interacts back on the beam.

For a beam traveling ultra-relativistically (essentially at the speed of light) then the Panofsky–Wenzel theorem \cite{10} relates the transverse component to the longitudinal wakefield. This formula is very convenient in beam dynamics simulations where the transverse momentum imparted to the beam is required. However, it is valid for a strictly ultra-relativistic beam. In the low-velocity regime, in a photocathode for example, velocity dependent corrections will be required. Here we drive these additional correction terms and obtain general relation between the transverse and longitudinal wakefield valid for non-relativistic and ultra-relativistic beams. These effects are particularly relevant in photocathodes where the initial acceleration of the beam takes place. Currently, there are several numerical computer codes available for modeling electron sources geometries which are able to include the effects of space charge fields. Among of these codes, we mention the classical code PARMELA \cite{11} which incorporates electrostatic space–charge effects and TREDI \cite{12} which incorporates e.m space charge forces using Lienard–Wiechart potentials in free space. These codes have been extensively used for simulating photoinjectors \cite{13}. There are several approximations employed in these codes. In particular, the accelerating cavities are assumed to be electromagnetically decoupled from each other. These approximations require the deflecting field (higher order modes fields within the cavity) to be localized and to not propagate along the accelerator. These modes are excited by the beam and their relative amplitudes are dependent on the temporal length of the beam and on the velocity of the beam. However, these computer codes do not in general take into account these effects. These assumptions are quite accurate when the beam is sufficiently far from the cathode. However, the underlying assumptions are invalidated in the immediate vicinity
of the cathode. This is indeed the case when we consider electrons at their initial stage of the emission from the photocathode. Furthermore, in many photoinjectors the length of the cavity is not negligible in comparison to that of the bunch. For example, for the ELBE [14] photocathode the beam is 2 mm long and the cavity is approximately 37.7 mm long. Finally we note that simulations using particle in cell (PIC) codes such as MAFLA [15,16] includes these space charge and wakefield effects. However, there is a limited analysis of wakefield effects from an analytical perspective and in the present work we develop equations for the e.m field in terms of the excitation current.

Recently, analytical techniques for computing e.m potentials in low frequency photoinjectors have emerged in which bunch acceleration is assumed constant [17]. Here we employ a similar technique to analyze the transverse momentum kick imparted to the beam in a photocathode cavity. We note that the effects of these e.m potentials or wakefields on the beam have been successfully studied in connection with the beam break up [3–7] in which the beam is assumed to be ultra-relativistic and is assumed to be essentially traveling with the velocity of light. Our analysis does not make this assumption as in photoinjectors the beam is traveling substantially less than the velocity of light and hence additional velocity-dependent effects must be taken into account.

We begin the analysis by following a time dependent formulation [18] for the ELSA photoinjector shown in Fig. 1 (where \( g \) is the cavity length, \( \gamma \) and \( r_0 \) are the hole and cavity radii, respectively). The ELSA facility is a high brightness 18 MeV electron source dedicated to electron irradiation, \( \gamma \)-rays and picoseconds hard and soft X-rays. It consists of a 144 MHz RF-photoinjector producing short bunches which are further accelerated to an energy varying from 2 to 18 MeV using two additional 433 MHz accelerators cavities [19].

An analytical expression for the transverse momentum imparted to the electrons inside the beam will be employed. The longitudinal fields are analyzed and related to the transverse fields by application of a modified form of the Panofsky–Wenzel theorem [10] which includes velocity-dependent effect. By applying the causality principle we are able to simplify the effects associated with the actual cavity, illustrated in Fig. 1, to an analysis of the e.m fields in a pill-box cavity. According to analytical and semi-empirical results [20,21] the exit hole influence can be neglected as long as \( r_0/\gamma < \frac{1}{\gamma} \). A first consequence of causality principle—which prevents the beam’s field influence at a distance from the emissive cathode greater than \( ct \)—is thus to restrict the part of the photoinjector walls able to contribute to the beam wake. For an RF-field on the cathode \( E_0 \) with an amplitude of a few tens of MV/m, it takes approximately 350 ps for the beam head to cross a photoinjector cavity having a gap \( g \) of say 6 cm. Within this time, the radial wall (\( r = \gamma g \), \( r < r_{\text{max}} \)) is not reached by the e.m signal emanating from the beam. This means that only a small fraction of the transverse wall (located \( z = 0 \), \( z = g \), \( r < r_{\text{max}} \)) can generate a wake which is able to influence the beam. Therefore, a pill-box cavity model is well-founded and is illustrated in Fig. 1, for \( \gamma > ct/g \) (\( c \) being the speed of light and \( t_p \) is the time at which the beam head reaches the photoinjector exit).

Panofsky and Wenzel considered the transverse momentum imparted to an ultra-relativistic particle moving parallel to the axis of the cavity with a velocity equal to that of light. If \( g \) is the length of the cavity, then the transverse momentum \( p_\perp \) is given

\[
p_\perp = \int_0^z F_\perp \, dz
\]

where \( F_\perp \) is the force exerted on the particle, \( e \) is the electron charge, \( E \) and \( B \) are the electric and the magnetic wakes, respectively, \( \vec{v} \) is the particle velocity, \( z \) is the distance along the axis and \( \perp \) denotes perpendicular components.

Since \( |\vec{v}| \approx c \), the particle direction remains unchanged by the transverse force. Panofsky and Wenzel have shown that the above equation can be simplified by expanding the right-hand side of it in terms of a vector and scalar potential [10]. Moreover, for an ultra-relativistic beam the scalar term is zero in this formulation. However, we consider beams which are not necessarily traveling ultra-relativistically and indeed initially the beam travels non-relativistically when it is emitted from the cathode. A rigorous description of this situation based on the normal mode analysis model in a pill-box cavity is the aim of this paper. The transverse momentum expression will be computed for the photoinjector schematized in Fig. 1. The RF-accelerating field will be assumed to be constant and we note that for ELSA photoinjector (which operates at 144 MHz), this is a good approximation provided the pulse duration \( \tau \ll 7 \) ns.

### 2. Derivation of transverse momentum kick for an accelerated electron beam

#### 2.1. Problem setup

The transverse momentum \( p_\perp \) imparted to a particle with a velocity \( \vec{\beta} \) (normalized with respect to the velocity of light) and an amount of charge \( Q \) at time \( t \) inside the beam accelerating in the \( z \)-direction through the RF cavity schematized in Fig. 1, is given as an integral over the duration of the force

\[
p_\perp(r,z,t) = \int_{t(z = z_0)}^{t(z = z_q)} F_\perp(r,z,t) \, dt = \quad \int_{t(z = z_0)}^{t(z = z_q)} \left[ E_\perp(r,z,t) + \left( \vec{\beta}(z,t) \vec{c} \times \vec{B}(r,z,t) \right)_\perp \right] \, dt
\]

where the initial limit of integration \( t(z = z_0) \) denotes the time at which the tails of the beam is located at longitudinal coordinate \( z_0 \), \( t_f(z) \) denotes the time for an element of the beam at location \( z \), \( Q(t) \) is the amount of charge included between \( z_0 \) and \( z \) in the time interval \( \Delta t = t_f - t_i \), \( \vec{\beta}(z,t) \) is the beam velocity. The beam velocity \( \vec{\beta}(z,t) \) and acceleration \( \gamma(z,t) \) are shown to be parallel to the accelerated field \( E_0 \) and independent of time [18]:

\[
\vec{\beta}(z,t) = \vec{\beta}(z) \vec{c}
\]

\[
\gamma(z) = \gamma(z) \vec{c}
\]

![Fig. 1. Schematic representation illustrating the essential geometry of the ELSA photoinjector (144 MHz cavity).](image-url)
with
\[ \beta(z) = \sqrt{\frac{(1 + Hz(t))^2 - 1}{1 + Hz(t)}} \] (4)
\[ \gamma(z) = 1 + Hz \] (5)
\[ z(t) = \frac{1}{H}(\sqrt{1 + (Hz(t) - tz_0)^2} - 1) \] (6)
\[ H^{-1} = \frac{mc^2}{eE_0} \] (7)
where \( m \) is the rest mass of the electron, \( z(t) \) is the longitudinal coordinate of electron at time \( t \), and \( tz_0 \) is the time at which element \( z \) of the beam leaves the photocathode.

Eq. (2) can be simplified by expanding the first and the second terms in the right-hand side in terms of the vector and scalar potentials \( \vec{A} \) and \( \Phi \), respectively. The first term can be written as
\[ E_z(r, z, t) = -\frac{\partial A_z}{\partial z} - \nabla \cdot \Phi(r, z, t) \] (8)
and the second term can be written as
\[ \vec{B}(r, z, t) = \nabla \times \vec{A} \] (9)
\[ (\beta \vec{c} \times \vec{B}) = (\nabla \times (\beta \vec{c} \cdot \vec{A}) - (\beta \vec{c} \cdot \nabla) \vec{A}) \] (10)
\[ (\beta \vec{c} \times \vec{B}) = \nabla (\beta \vec{c} \cdot \vec{A}) - (\beta \vec{c} \cdot \nabla) \vec{A} \] (11)

The vector and scalar potential \( \vec{A}(r, z, t) \) and \( \Phi(r, z, t) \) are solutions of Maxwell’s equation in a Coulomb gauge in the axisymmetric cylindrical cavity region (pill-box) \( 0 \leq z \leq g; 0 \leq r \leq \pi \) with the appropriate boundary conditions in a conducting circular pipe:
\[ \nabla^2 \Phi = -\frac{\rho}{\varepsilon_0} \] (12)
\[ \square \vec{A} = \mu_0 \beta \frac{\partial}{\partial t} (\nabla \Phi) \] (13)
where \( \nabla^2 \) is the Laplacian in cylindrical coordinates, \( \square \) is the d’Alembertian operator, \( \rho \) is the charge and current density, respectively:
\[ \rho(r, z, t) = \frac{\omega}{\pi \alpha^2 a^2} [1 - \Theta(r - a)] \] (14)
describe a radially uniform beam of radius \( a \) and velocity \( v(z) = \beta(z)c \), carrying a total current \( I \). Its axial current profile \( \omega(z, t) \) is uniform, with stiff front and back, and time length \( \tau \). \( \Theta(r - a) \) is the Heaviside function.

2.2. Initial and boundary conditions

For points on the conducting transverse wall at the location \( z = 0 \) (where the cathode is located; the initial conditions are prescribed according to those at the beginning of electron emission, \( t = 0 \), and we assume \( \Phi(r, z, t) = 0 \), \( \vec{A}(r, z, t) = 0 \); \( \partial \Phi/\partial t = 0 \) and \( \partial \vec{A}/\partial t = 0 \). At the location of the anode \( z = g \), the potential is zero because this surface is grounded. Therefore, we are required to satisfy Dirichlet conditions for \( \Phi \) and the tangential component of \( \vec{A} \); \( \Phi(r, z = 0, t) = 0 \) , \( \vec{A}(r, z = g, t) = 0 \), and Neumann conditions for the normal component of \( \vec{A} \); \( \partial \vec{A}(r, z = g, t)/\partial n = 0 \).

According to symmetries, the only non-vanishing components, in cylindrical coordinates, are \( E_z \), \( E_r \), and \( B_\theta \).

2.3. Analysis of potentials

These potentials are expressed as infinite series of the product of the eigenmodes of a pill-box cavity as described in [22]
\[ \Phi(\vec{x}, t) = \sum_{n \neq 0} \alpha_n(t) \phi_n(\vec{x}) \] (15)
\[ \vec{A}(\vec{x}, t) = \sum_{n \neq 0} \vec{a}_n(t) \vec{\phi}_n(\vec{x}) \] (16)
where
\[ \nabla^2 \phi_n + \frac{\omega_n^2}{c^2} \phi_n = 0 \] (17)
\[ \nabla^2 \vec{a}_n + \frac{\omega_n^2}{c^2} \vec{a}_n = 0 \] (18)
\[ \omega_n = \sqrt{\left( \frac{J_n}{\lambda_n} \right)^2 + \left( \frac{\pi p}{g} \right)^2} \] (19)
where \( J_n \) is the \( n \)th zero of Bessel function \( J_n \); \( \lambda_n = n, p; n = 1, 2, 3, \ldots \) is radial mode number, and \( p = 0, 1, 2, \ldots \) is axial mode number.

On the other hand, \( \phi_n(\vec{x}) \) and \( \vec{a}_n(\vec{x}) \) are also known for the standard pill-box cavity [22];
\[ \phi_n(\vec{x}) = \phi_{n0} \left( \frac{r}{\lambda_n} \right) \sin \left( \frac{\pi p}{g} z \right) \] (20)

Substituting Eqs. (15) and (16) into Eqs. (8) and (9), respectively, we obtain
\[ \vec{E}(r, z, t) = -\sum_n \left( \frac{\partial \phi_n}{\partial t} \vec{a}_n(r, z) + \alpha_n(t) \nabla \phi_n(r, z) \right) \] (21)
\[ \vec{B}(r, z, t) = \sum_n (q_n(t) \nabla \times \vec{a}_n(r, z)) \] (22)
Making use of Eqs. (21) and (22) in Maxwell’s equations and using the characteristic orthogonality properties of the modes enables the driven equation of simple harmonic motion to be obtained for mode \( q_n(t) \) [23]
\[ \frac{\partial^2 q_n(t)}{\partial t^2} + \omega_n^2 q_n = \frac{1}{2U_s} \int \vec{j} \bullet \vec{a}_n d^3x \] (23)
\[ \alpha_n = \frac{1}{2 T_s} \int \rho \phi_n d^3x \] (24)
\[ U_s = \frac{1}{\lambda_n} \int \frac{d^3x}{a^2} \] (25)
where driving terms on the right-hand side of Eq. (23) and \( U_s \) and \( T_s \) are given by Salah et al. [18] as
\[ U_s = \frac{\pi \mu_0 \hbar^2 g^2 a^2 \phi_n^2 (U_s)}{4} \] (26)
\[ T_s = \frac{\hbar^2}{2} U_s \] (27)
We note that, setting the right-hand side (i.e. the driving term) to zero results in the characteristic eigenvalues, \( \omega_n \), to be obtained as a solution to the modal frequencies. Furthermore in the driven case, by means of Eq. (21), the first term in the right-hand side of
Eq. (2) becomes:
\[ E_t(r, z, t) = -\sum_j \left( \frac{\partial q_j}{\partial t} a_{ij}(r, z) + z_j(t) \frac{\partial \phi_j(r, z)}{\partial r} \right) \] (26)
while the second term becomes
\[ \beta c \times \mathbf{B}_0 = \beta c \sum_j q_j(t) \left( \frac{\partial a_{ij}}{\partial r} \frac{\partial q_j}{\partial z} - \frac{\partial a_{ij}}{\partial z} \frac{\partial q_j}{\partial r} \right). \] (27)

2.4. Analytical expression of the transverse momentum

A rigorous expression for the transverse momentum imparted to the electrons inside the beam can be obtained by substituting Eqs. (26) and (27) into Eq. (2) from which we obtain
\[ P_{\perp}(r, z, t) = Q(t) \int_0^{t_f} \left[ \sum_j \left( -\frac{\partial q_j}{\partial t} a_{ij}(r, z) - z_j(t) \frac{\partial \phi_j(r, z)}{\partial r} \right) + \beta \epsilon q_j(t) \left( \frac{\partial a_{ij}}{\partial r} \frac{\partial q_j}{\partial z} - \frac{\partial a_{ij}}{\partial z} \frac{\partial q_j}{\partial r} \right) \right] \, dr. \] (28)

By means of Eq. (20), Eq. (28) becomes
\[ P_{\perp}(r, z, t) = Q(t) \int \left( \sum_j \left( \frac{\partial q_j}{\partial t} \frac{\pi p}{\beta q_j} a_0 \right) + z_j(t) \left( \frac{\beta}{\epsilon q_j} \right) \sin \left( \frac{\pi p}{\beta} z(t) \right) + \beta \epsilon q_j(t) \alpha \pi a_0 \cos \left( \frac{\pi p}{\beta} z(t) \right) \right) \, dr. \] (29)
The integral form of the transverse momentum in Eq. (29) requires integration over rapidly oscillating terms in Fourier k-space.

3. Results and discussions

We now apply the analysis developed in the previous section to obtain transverse momentum kick imparted to the beam in the ELSA photoinjector. The analytical expressions for the transverse momentum are evaluated, with a FORTRAN program written expressly for this purpose together with the statistical library ISML. An appropriate parameter set for a representative sample of momentum maps for the ELSA photoinjector is: \( l = 100 \ A, \pi a^2 = 1 \ cm^2 \) (\( a \) is the radius of the beam) \( E_0 = 30 \ MV/m \) and \( \tau = 30 \ ps \).

The evolution of the transverse momentum imparted to electrons on the beam axis as a function of time \( T = Hct \) and \( r \) at the end of the photoemission (i.e. at the instant at which the head of the beam is located at \( r = \tau = 30 \ ps \)) is illustrated in Fig. 2. From this figure we conclude that the momentum imparted to the electrons is in the positive radial direction. Furthermore, it is more intense for electrons at the head of the beam and decreases towards the tail. Moreover, the momentum kick on the head of the beam is, as expected an increasing function of \( r \). This increasing kick is due to the strong contribution of the cathode wakefield. Indeed, a multipole expansion [1] of the e.m field clearly indicates higher order multipoles will become more important as the offset becomes larger from the axis of the cavity. The figure also indicates that electrons at the tail of the beam are accelerated longitudinally by the wakefield and this is driven by the electrons situated at the head of the beam.

Fig. 3 repeats Fig. 2, but for the instant at which the head of the beam is located at \( r = t_0/4 \), where \( t_0 = 250 \ ps \) is the time at which the head of the beam reaches the exit of the photoinjector. In this case, whilst electrons situated at the tail of the beam continue to be accelerated in the longitudinal direction, those electrons not situated in the immediate vicinity of the tail gain momentum in the negative radial direction. The electrons located away from the head of the bunch are affected by the radial components of the wakefield driven by the beam. Again, the wakefield is more intense at the head of the beam and decreases rapidly towards the tail.

Figs. 4 and 5 illustrate the evolution of the transverse momentum imparted to the electrons within the beam at the instant at which the head of the beam is located at \( t = t_0/4 \) and \( t = 3t_0/4 \), respectively. From these two figures, we may conclude the following:

- There is a region situated at the center of the beam slightly towards the head, in which the electrons have not imparted any transverse momentum. However, the length of this region decreases as one moves out from the center of the beam. The length of this region becomes clear at the instant \( t = 3t_0/4 \) as illustrated in Fig. 5. Moreover, this region moves progressively towards the head as one can see from inspection of Figs. 4 and 5.
There is a strong asymmetry in the momentum imparted to the electrons within the beam with respect to the region in which the momentum kick is zero. Electrons situated to the left of this region gain momentum in the positive radial direction, whereas, those situated to the right gain momentum in the negative radial direction. We postulate that this asymmetry may be necessary in order to keep an equilibrium distribution of the electrons within the beam during acceleration.

Fig. 6 represents the special case of the momentum imparted to electrons within the beam at the instant at which the head of the beam is located at $t=t_{ge}$. It confirms the fact that there is a region in which the electrons are imparted a constant radial momentum towards the head of the bunch. The length of this region decreases slightly as one move from the center to the edge of the beam. Moreover, this region is extended from the head to the tail of the beam provided electrons are located near the axis of symmetry. Electrons situated in this region of the beam rapidly become relativistic. We may expect to see this behavior for any radial offset as the beam crosses additional cavities and the beam becomes ultra-relativistic.

4. Conclusion

We have analyzed and simulated the time-dependent evolution of the transverse momentum imparted to the electrons along a bunch for a short-pulsed intense electron beam emitted from a cathode used in a RF-FEL photoinjector. Thus, this work should help facilitate understanding fundamental design issues involved in high-brightness electron sources. This has particular relevancy for systems requiring high-brightness electron sources such as the next generation of linear colliders (the ILC [24,25] in particular) and for XFEL [26,27] and ERL [28,29] synchrotron radiation sources. We have quantitatively analyzed the interaction of the wakefield within an electron bunch in a photocathode. The analysis provides analytical formulae which may provide a useful tool in aiding the design of future photocathodes. This can be achieved by minimizing the transverse momentum imparted to the electrons within the beam by minimizing the beam’s wakefield. The later depends on the dimensions of the beam, the pulse duration of the beam, the acceleration field, and the geometry of the photoinjector.

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