Analysis of space charge fields using Lienard–Wiechert potentials and the method of images in the RF-free electron laser photoinjectors

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ABSTRACT

Based on Lienard–Wiechert retarded potentials and the potential due to a beam-induced image charge on the cathode, a rigorous relativistic description of a beam transport inside an RF-photoinjector is presented. Velocity-dependent effects are explicitly taken into account. Simulations are presented for parameters appropriate to the ELSA photocathode where velocity-dependent effects are particularly important. These simulations reveal that at the center of the cathode (r = 0, z = 0) the beam’s self-field and the field driven by the charges image on the cathode are equal. However, the self-field of the beam is dominated by the field due to the charges image on the cathode as one moves from the tail to the head of the beam. Far from the cathode, the self-field becomes the dominant one.

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1. Introduction

RF-photoinjectors facilitate the production of high-quality beams, with both low emittance and low-energy spread. Once charged particles are emitted from the cathode, they are rapidly accelerated. During this process, the bunches of charged particles excite parasitic higher-order modes within the cavity which have the ability to modify the acceleration of particles within the beam.

We contrast the beam dynamics interaction in a photoinjector with that which occurs when the particles have been accelerated to ultra-relativistic energies. At these ultra-relativistic energies the electromagnetic field (e.m.) excited by the charged-particle beam is conveniently represented as a wakefield [1] and the particles are assumed to be traveling at the velocity of light. This representation is valid for the acceleration of ultra-relativistic beams in the main linacs of linear colliders and light sources for example [2]. The wakefield excited in this regime has been analyzed in some detail [3] and several codes are available to simulate both the wakefield effects and the associated beam dynamics of the interaction [4–6]. However, the initial stage of acceleration constitutes a different regime, and warrants a treatment in which both velocity-dependent and retardation effects must both be taken into account explicitly. This clearly is the case in photoinjectors where the energy is sufficiently low such that a rapid change in particle velocity occurs during acceleration. In this regime space-charge forces and self-field effects are important issues to consider in an analysis of the beam dynamics [1,7–9].

Wakefield effects in photoinjectors have been considered from the perspective of a normal-mode expansion [10–13]. In this paper the e.m field is analyzed in terms of Lienard–Wiechert potentials and the method of images. An important technique in the realization of high-brightness electron injectors is the velocity (or klystron bunching) in which the acceleration is off crest and there is significant rotation of the electron bunch in longitudinal phase space. Our model assumes that slices of the electron beam cannot overtake one another. Moreover, the restriction of a constant velocity beam is removed [1–3].

This technique is applied to parameters of the ELSA photoinjector facility [14]. A schematic representation of the essential parameters of this photoinjector cavity is illustrated in Fig. 1. We are able to simplify this analysis to that of the modes in a pill-box cavity by considering the time required for the radiation field to propagate to the walls of the cavity. The e.m field propagates at c (the speed of light), and after a time t, the field has propagated to a distance ct. Thus, by the principle of causality, at distances larger than ct the field excited by the beam is unable to influence subsequent parts of the beam. If we consider the ELSA parameters,
for an RF-field on the cathode $E_0$ with amplitude of a few tens of MV/m, it takes approximately 350 ps for the head of the beam to traverse a photoinjector’s cavity of gap length $g$–6 cm. Within this time, the radial wall ($r = 50$ cm) is not reached by the electromagnetic signal emanating from the beam. This means that only a small fraction of the transverse wall (located $z = 0$, $z = g$, $t < t_{max}$) can generate a wake which is able to influence the beam. Therefore, a pill-box cavity model can be used to represent the modes excited, provided $\omega > ct_g$ (where $t_g$ is the time at which the head reaches the photoinjector exit) [15]. Also, analytical and semi-empirical results [16,17] indicate that the influence of the exit hole can be neglected as long as $r_0/(g R ) \ll (1/3)$, where $r_0$ and $g$ and are the hole and cavity radii, respectively (Fig. 1).

This paper is organized as follows: The next section delineates the fundamental zero-order equations of motion, and the section following that focuses on an analysis of the Lienard–Wiechert fields excited by individual charged particles within the electron beam. The penultimate section develops the global field excited by a specific current distribution of particles. The final main section concludes with a set of e.m. fields excited in a cavity with parameters similar to that of ELSA. Results are also compared with those obtained from a previous independent analysis and are found to be in excellent agreement with those produced from an analysis in this paper.

2. Zero-order equations of motion

The electron beam is emitted as a pulse from the photocathode in the time interval $t = 0$ to $\tau$ (where $\tau$ is the time at which the photoemission ends). It is assumed to be axisymmetric, of radius $a$, and with a constant and uniform current density $J$. The acceleration RF-electric field $E_0$ can be considered to be constant and uniform provided the pulse duration is restricted according to $\tau \ll 1/\nu$ (where $\nu$ is the RF frequency) and the beam radius $a$ is sufficiently small compared to the cavity radius $g$. For the ELSA photoinjector $\nu = 144$ MHz, $g = 60$ cm, $\pi a^2 = 1$ cm$^2$, and $\tau \ll 7$ ns; both of these conditions ensure the assumption of uniformity is justified. Furthermore, under these conditions, the beam velocity $\beta(z, t)$ and acceleration $\eta(z, t)$ are shown in [10] to be parallel to the accelerated field $E_0$ and independent of time:

$$
\beta(z, t) = \beta(z) \hat{u}_z \\
\eta(z) = \eta(z) \hat{u}_z
$$

with

$$
\beta(z) = \frac{\sqrt{(1 + Hz(t))^2 - 1}}{1 + Hz(t)} \quad (2)
$$

$$
\eta(z) = 1 + Hz \quad (3)
$$

$$
z(t) = \frac{1}{\Pi} \left( \sqrt{1 + (Hz(t - t_z))^2} - 1 \right) \quad (4)
$$

$$
H^{-1} = \frac{m c^2}{e E_0} \quad (5)
$$

3. Lienard–Wiechert electromagnetic fields of a point charge

3.1. Definition

The electromagnetic fields ($\vec{E}, \vec{B}$) generated at time $t$ and point $P$, by an electron, moving on a specified trajectory depends on the retarded position $W(t')$ of the electron at time $t'$ (Fig. 2). These fields are driven from scalar and vector potentials $\Phi$ and $\vec{A}$, respectively. Taking into account the boundary condition imposed on the cathode by the equipotential and by the principle of causality, these fields are given by the characteristic Lienard–Wiechert expression as

$$
\vec{E}(P, t | W) = -\frac{e}{4\pi\varepsilon_0} \left( \frac{\hat{R} - \hat{\beta}(t') \hat{R}}{\gamma(t')(R - \hat{R} \cdot \hat{\beta}(t'))^3} \right) \left( \frac{\hat{R} \times (\hat{R} - \hat{\beta}(t') \hat{R}) \times (\hat{R} \beta(t') / \partial t')}{c(R - \hat{R} \cdot \hat{\beta}(t'))^3} \right) \quad (6)
$$

$$
\vec{B}(P, t | W) = \frac{1}{c} \frac{\hat{R}}{R} \times \vec{E}(P, t | W) \quad (7)
$$

with

$$
\gamma(t') = \frac{1}{\sqrt{1 - \beta(t')^2}} \quad (8)
$$

where $\varepsilon_0$ is the permittivity of free space and all other parameters are as indicated in Fig. 2. The first term in the parenthesis in Eq. (6) is the velocity field while the second is the acceleration or retardation field. The former corresponds to energy transfer in the vicinity of the beam as it decays radially as $1/R^2$, whilst the latter gives rise to a finite energy transfer in the radiation zone with a field which decays with a $1/R$ dependence.

![Fig. 2. Field driven by an electron moving on a specific trajectory.](image-url)
3.2. Coordinates system and fields projection

The components of the electromagnetic field driven by an electron within the beam and the charge image on the cathode is obtained by a projection of Lienard–Wiechert fields given by Eqs. (6) and (7) on the axes illustrated in Fig. 3. We perform this projection in the laboratory frame. The point at which we observe the field is taken as the origin of the frame. Cylindrical coordinates \((s, \theta, z)\) of \(W\) are defined in Fig. 3. The vector from the retarded position of the electron \(W(t')\) to the field point \(P\) is

\[
\begin{vmatrix}
-s \cos \theta \\
-s \sin \theta \\
 z - \zeta
\end{vmatrix}
\]

where the superscript \(^\cdot\) denotes values taken at time \(t'\).

In the paraxial approximation used in the beam dynamics analysis, the velocity \(\mathbf{v}(t)\) and the acceleration \(\mathbf{a}(t)\) of the beam are assumed to be collinear. Therefore, the double-cross product in Eq. (6) may be evaluated as

\[
\mathbf{R} \times \left( \mathbf{r} \times \frac{d}{dt} \mathbf{R} \right) = \frac{1}{R^2} \left| \begin{vmatrix}
\frac{d\mathbf{R}}{dt} & \frac{d}{dt} (z - \zeta) \cos \theta \\
\frac{d}{dt} (z - \zeta) \sin \theta & \frac{d\mathbf{R}}{dt} \\
\end{vmatrix} \right|
\]

Eqs. (10) and (6) enable the electric field components to be obtained on the axes given in Fig. 3

\[
\begin{align*}
E_{z,\beta}(P, t|W) &= \frac{e}{4\pi \varepsilon_0 c}\frac{\zeta - z + \beta \sqrt{s^2 + (\zeta - z)^2}}{\sqrt{s^2 + (\zeta - z)^2 + \beta'(\zeta - z)^2}} \\
E_{z,\beta}(P, t|W) &= \frac{e}{4\pi \varepsilon_0 c}\frac{\beta' s^2}{\sqrt{s^2 + (\zeta - z)^2 + \beta'(\zeta - z)^2}} \\
E_{r,\beta}(P, t|W) &= \frac{e}{4\pi \varepsilon_0 c}\frac{s \cos \theta}{\sqrt{s^2 + (\zeta - z)^2 + \beta'(\zeta - z)^2}} \\
E_{r,\beta}(P, t|W) &= \frac{e}{4\pi \varepsilon_0 c}\frac{s \sin \theta}{\sqrt{s^2 + (\zeta - z)^2 + \beta'(\zeta - z)^2}}
\end{align*}
\]

(11)

(12)

(13)

(14)

\[
E_{\theta,\beta}(P, t|W) = \frac{e}{4\pi \varepsilon_0 c}\frac{\beta'(\zeta - z) s \cos \theta}{\sqrt{s^2 + (\zeta - z)^2 + \beta'(\zeta - z)^2}}
\]

(15)

where the indices \(\beta = \beta / c\) denote the field components dependent on velocity and acceleration, respectively.

3.3. Field due to charges image on the cathode

The charges image on the cathode can be represented by a set of symmetric charges \(+e\) at each instant \(t\) with respect to the cathode. The field component due to these images can be obtained by replacing \(-e\) with \(+e\), with \(\beta = -\beta\), and with \(\zeta = -\zeta\).

4. Domain of retarded sources

In order to derive a general expression for the fields produced by electrons within the beam and the corresponding charges image on the cathode, we must determine the effective domain of the retarded sources. This assigns the region of the electrons occupy within the beam and the corresponding image occupied by the total field.

4.1. Instant \(t\) in the plane of abscissa \(\zeta\)

Fig. 4 illustrates the plane of abscissa \(\zeta\). This figure is used to determine the ensemble of points \((s, \zeta)\) in the plane from which an electromagnetic signal is emitted at time \(t\) and arrives at point \(p\) at time \(t\). The emission of this signal assumes that there are a finite number of electrons in the plane at abscissa \(\zeta\). This assumption can be justified as follows: for \(z\) given by Eq. (4), and taking \(t_c = 0\) in order to determine the time \(t\) at which the head of the beam reaches the plane under consideration. Hence

\[
t = \frac{1}{c} \sqrt{\zeta \left( \zeta + \frac{2}{\rho} \right)}
\]

(17)

Moreover, the time required for the signal to move from a point in the plane of abscissa \(\zeta\) to the point \(P\) is greater than \((|\zeta| - z)/c\). This analysis imposes

\[
\frac{1}{c} \sqrt{\zeta \left( \zeta + \frac{2}{\rho} \right)} \leq t' \leq t - \frac{|\zeta - z|}{c}
\]

(18)

However,

\[
t' = t - \frac{WP}{c} = t - \frac{1}{c} \sqrt{s^2 + (z - \zeta)^2}
\]

(19)

Since \(s\) is in the plane of abscissa \(\zeta\), it must satisfy the following inequality:

\[
s \leq s_{\text{max}} = \sqrt{\left( \frac{ct}{\sqrt{\zeta \left( \zeta + \frac{2}{\rho} \right)}} \right)^2 - (\zeta - z)^2}
\]

(20)
This inequality only exists if
\[ ct - \sqrt{\left(\frac{\zeta}{c} + \frac{2}{H}\right)^2} \geq |\zeta - z| \]  
(21)

Therefore, we have two cases to consider

1. \( \zeta \leq z \)

The last equation gives
\[ \zeta \leq \frac{(ct - z)^2}{2(H^{-1} + z - ct)} \]  
(22)

By means of Eqs. (17) and (22) and after some algebraic simplifications, we obtain
\[ \frac{(ct - z)^2}{2(H^{-1} + z - ct)} - z \geq 0 \]  
(23)

Since the difference in the above equation is greater than zero, we conclude that for all planes described by the abscissa \( \zeta \leq z \), there exist electrons which emit electromagnetic radiation arriving at point \( P \) at time \( t \).

We now consider the opposite limit.

2. \( \zeta \geq z \)

Following the same procedure outlined previously, we obtain the following inequality:
\[ \zeta \leq \frac{(ct + z)^2}{2(H^{-1} + z + ct)} = \zeta_{\text{max}} \leq \zeta_f \]  
(24)

where \( \zeta_f \) corresponds to the head of the beam.

At instant \( t \), all electrons have a longitudinal coordinate within \( \zeta_f \). Therefore, \( \zeta_{\text{max}} \leq \zeta_f \) and \( \zeta_{\text{max}} \) exist if and only if \( \zeta \leq \zeta_{\text{max}} \).

4.2. Instant \( t^2 \) in the plane of abscissa

Fig. 5 shows the plane of abscissa \( \zeta \) and we use this figure to determine the ensemble of points (\( s, \zeta \)) in the image space from which an electromagnetic signal is emitted at time \( t^2 \) and arrives to point \( P \) at time \( t \). The emission of this signal assumes that there are a finite number of electrons in the plane of abscissa \( \zeta \). We justify this assumption by considering the arrival time of the beam’s head at time \( t \) to plane \( \zeta \):
\[ t = \frac{1}{c} \sqrt{\left(-\frac{\bar{\zeta}}{c} + \frac{\bar{\zeta}^2}{H}\right)} \]  
(25)

Following the same analysis given in Section 4.1, we can readily show that in the plane of abscissa \( \zeta \), \( s \) must satisfy the following inequality:
\[ s \leq \zeta_{\text{max}} = \sqrt{\left(\frac{ct - \sqrt{\bar{\zeta} - \frac{\bar{\zeta}^2}{H}}}{2}\right)^2 - (\bar{\zeta} - z)^2} \]  
(26)

Since \( \bar{\zeta} \leq z \), Eq. (26) imposes
\[ \bar{\zeta} \leq \bar{\zeta}_{\text{max}} = \frac{(ct - z)^2}{2(H^{-1} + z - ct)} \]  
(27)

At instant \( t \), we have \( \bar{\zeta} \leq \bar{\zeta}_{\text{max}} \). Therefore, from symmetry considerations, \( \bar{\zeta} \geq \bar{\zeta}_{\text{max}} \) and \( \zeta_{\text{max}} \) exist if and only if \( \bar{\zeta} \leq \bar{\zeta}_{\text{max}} \).

Thus, we categorize an ensemble of electrons and their corresponding images by a domain of retarded sources. This region is a cylinder of radius \( r \) and length \( L(t) \) whose axis of symmetry passing by \( P \) and parallel to the axis of a cylinder whose meridian section satisfies the equation \( s = z_{\text{max}}(\bar{\zeta}) \). For the charges image, it is sufficient to replace \( L(t) \) with \( -L(t) \). It is from this volume that any observable at \( P \) witnesses the incoming radiation.

5. Generation of global fields from the individual ones

Here, we generate the global fields driven by the beam using the field components driven by an individual electrons and their charges image. For the sake of simplicity, we limit the analysis to analyze how the longitudinal component \( E_z \) of the global field is generated (other components are generated using a method which is formally identical to that used for the longitudinal one). To this end, we consider a cylindrical-beam pulse, with radius \( a \), carrying a current \( I \). This current is emitted from the cathode with a constant and radially uniform charge and current density \( \rho(r, z, t) \) and \( j(r, z, t) \), respectively, and they are given as
\[ \rho(r, z, t) = \frac{I(\bar{\zeta}(z, t))}{\pi a^2} \]  
(28)
\[ \bar{j}(r, z, t) = \frac{\sigma(\bar{\zeta}(z, t))}{\pi r a} \]  
(29)

Here \( \bar{j}(z) = \sigma(\bar{\zeta}(z)) \) denotes the time-independent velocity, \( \bar{\zeta} \) is the total current whose axial profile \( \bar{\zeta}(z, t) \) is uniform (with a stiff front and back, and time duration \( t \)) and \( \Theta(r - a) \) is the Heaviside-step function.

If \( n(W, t) \) is the density of electrons or charges image at time \( t \), and the boundary condition imposed at the cathode corresponds to an equipotential, then the longitudinal component of the global field at the point \( P \) is given by
\[ E_z(P, t) = \int d \bar{W} n(W, t) E_z(P, t|W) d^4W \]  
(30)

where \( E_z(P, t|W) \) and \( E_z(P, t|\bar{W}) \) are the field components due to an electron and charge image, respectively. Also, \( D \) and \( \bar{D} \) represent an ensemble of electrons and charges image, respectively, having an antecedent at retarded time \( t' \) and \( t'' \). Zero-order equations of electron motion result in analysis of a stationary system, i.e., the factors which characterize the system are explicitly time independent. Moreover, the paraxial approximation is used for the beam dynamics and thus in this case the electrons are displaced parallel to the axis of symmetry. Under these conditions, the beam velocity and acceleration are independent of time. As a consequence, this density function depends only on the longitudinal position:
\[ n(W, t) = n(\bar{\zeta}) = \frac{|P|}{e} = \frac{1}{e \beta c} \]  
(31)

The components \( E_z(P, t|W) \) and \( E_z(P, t|\bar{W}) \) can be written in terms of \( W(t) \) and \( \bar{W}(t) \)
\[ W(t') = \mathcal{J}_{2,d}(M) = \begin{cases} s' = s \\ \theta' = \theta \\ \zeta' = f(s, \theta, \zeta) \end{cases} \]  
(32)
\[ W(t') = \tilde{S}_2(M(t)) = \begin{cases} S' = S & \\
\theta' = \theta & \\
\xi = f(s, 0, \tilde{\xi}) & \end{cases} \]  \hspace{1cm} (32)

Hence,

\[ E_2(P, t) = \int_{W(t)} n(W, t) E_2(P, t|W) \tilde{S}_2(W) d^3W \]
\[ + \int_{W(t)} n(W, t) \tilde{E}_2(P, t|W) \tilde{S}_2(W) d^3W \]  \hspace{1cm} (33)

Since the integral is carried out with respect to \( W = W(t') \) and \( \tilde{W} = \tilde{W}(t') \), we obtain

\[ d^3W = \Omega(3^{-1}) d^3\tilde{W} \]  \hspace{1cm} (34)

\[ d^3\tilde{W} = \tilde{\Omega} (3^{-1}) d^3W \]  \hspace{1cm} (35)

with

\[ \Omega(3^{-1}) = \frac{\bar{\beta}}{\bar{\beta}} \left( 1 - \frac{\bar{\beta}(z - \tilde{\xi})}{\sqrt{s^2 + (z - \tilde{\xi})^2}} \right) \]  \hspace{1cm} (36)

\[ \tilde{\Omega}(3^{-1}) = \frac{\bar{\beta}}{\bar{\beta}} \left( 1 - \frac{\bar{\beta}(z - \tilde{\xi})}{\sqrt{s^2 + (z - \tilde{\xi})^2}} \right) \]  \hspace{1cm} (37)

where \( \Omega(3^{-1}) \) and \( \tilde{\Omega}(3^{-1}) \) are the Jacobians of \( 3^{-1} \) and \( \tilde{3}^{-1} \), respectively.

Eqs. (31)–(37) enable Eq. (33) to be written as

\[ E_2(P, t) = e \frac{1}{4\pi\varepsilon_0} \int_{D(P, t)} \frac{\zeta - z + \beta \sqrt{s^2 + (z - \tilde{\xi})^2}}{\sqrt{s^2 + (z - \tilde{\xi})^2 + \beta(c' - z)}} \]
\[ + \frac{\bar{\beta}^2}{c\sqrt{s^2 + (z - \tilde{\xi})^2 + \beta(c' - z)}} \cos \theta ds d\theta d\tilde{\xi} \]
\[ + \frac{e}{4\pi\varepsilon_0} \int_{D(P, t)} \frac{\zeta - z + \beta \sqrt{s^2 + (z - \tilde{\xi})^2}}{\sqrt{s^2 + (z - \tilde{\xi})^2 + \beta(c' - z)}} \cos \theta ds d\theta d\tilde{\xi} \]
\[ + \frac{\bar{\beta}^2}{c\sqrt{s^2 + (z - \tilde{\xi})^2 + \beta(c' - z)}} \cos \theta ds d\theta d\tilde{\xi} \]  \hspace{1cm} (38)

where \( J = e\beta\eta(\tilde{\xi}) \) is the current density and \( D(P, \zeta, t) \) denotes a disc located within the beam and constrained to be no larger than \( s_{\text{max}} \).

Following a similar treatment, components \( E_3 \) and \( B_0 \) are readily obtained:

\[ E_3(P, t) = \frac{1}{4\pi\varepsilon_0} \int_{D(P, t)} \frac{\sqrt{s^2}}{\sqrt{s^2 + (z - \tilde{\xi})^2 + \beta(c' - z)}} \cos \theta ds d\theta d\tilde{\xi} \]
\[ + \frac{\sqrt{s^2}}{\sqrt{s^2 + (z - \tilde{\xi})^2 + \beta(c' - z)}} \cos \theta ds d\theta d\tilde{\xi} \]
\[ + \frac{\sqrt{s^2}}{\sqrt{s^2 + (z - \tilde{\xi})^2 + \beta(c' - z)}} \cos \theta ds d\theta d\tilde{\xi} \]  \hspace{1cm} (39)

**6. Results and discussions**

Fig. 6 shows the global-axial electric field as a function of \( Z = Hz \) at the end of photoemission and at this instant the whole beam is extracted from the cathode (i.e. at the instant \( t = \tau = 30 \) ps) for the following parameters: \( I = 100 \text{ A}, \pi a^2 = 1 \text{ cm}^2 \) (\( a \) is the radius of the beam) and \( E_0 = 30 \text{ MV/m} \). This field bears comparison to that due to the space charge or self-fields and the charge image on the cathode. At the center of the cathode \( (r = 0, z = 0) \) the beam's self-field and the field driven by the charges image on the cathode are equal. This figure reveals that the beam's self-field is dominated by the field due to an image of charges on the cathode. If we only allow the effect of the self-field on the electrons within the beam, then it is clear that electrons situated between center of the beam and the rear of the beam are accelerated by the field from electrons situated in the head of the beam. In this case, the field driven by the charges image is a deceleration field. However, the global field indicates that electrons in the head of the beam accelerate those in the tail.

Fig. 7 repeats Fig. 6 but in this case for a radial field \( E_r \) at \( r = 2a/\sqrt{s} \). The electrons are accelerated in the radial direction by the global field. This acceleration becomes stronger as one moves from the tail to the head of the beam. Moreover, the self-field and the field due to the charges image have approximately the same magnitude in the region located from the beam's center and the rear of the beam. These two fields are of opposite polarity.

Fig. 8 repeats Fig. 7 but in the case the azimuthal component of the magnetic field \( B_0 \) is illustrated and it is the dominant component of the field.

Fig. 9 shows the axial global electric field \( E_z \) within the beam at time \( t = t_0/2 \) and \( r = a \) where \( t_0 \) is the time at which the head of the beam reaches the exit aperture. This field is compared to the self or space-charge field and the field due to the charges image on the cathode. At this instant, the beam travels downstream and far from the cathode so the influence of the cathode is less significant. However, in this figure it is clear that the field due to the charges image on the cathode is approximately constant as one moves from the tail to the head of the beam. This can be explained by bearing in mind that when the beam is far from the cathode, the field driven by the beam's charge image on the cathode reduces to that of the electric field due to a point charge. Therefore, at large distance from cathode, the beam's charge image appears to have diminished effectively to zero in the beam frame. This result implies that an electrostatic calculation of the field due to the cathode in the beam frame is sufficient provided the beam is far from the cathode.
away from the cathode. On the other hand, Fig. 9 shows that the self-field is the dominant one.

Fig. 10 illustrates the radial fields component $E_r$ at time $t = t_g/2$ and $r = a$, and shows that the self-field dominates that due to the charges image. The latter vanishes as soon as the beam travels sufficiently far from the cathode. The self or space-charge field accelerates the electrons within the beam in the positive radial direction. This acceleration is strongest at the center of the beam and decreases towards the beam’s head and tail.

Fig. 11 illustrates the azimuthal magnetic fields component $B_y$ at time $t = t_g/2$ and $r = a$, and this provides a confirmation that the self or space-charge field is the dominant one provided the beam travels far enough from the cathode. This field rotates in the positive azimuthal. This rotation is strongest at the center and decreases towards the edges of the beam. Moreover, the magnetic field due to the charges image on the cathode is of constant magnitude over the beam.

In order to check the validity of our model, the field maps obtained by the model described in this paper are compared to those that we recently published for the photoinjector ELSA where the analytical expressions for the $(E_B)$ $(x,t)$ maps have been obtained using a time-dependent normal mode analysis [10]. Direct comparisons of the resulting analytical expressions are difficult and hence we resort to numerical comparisons of the two approaches. The results of this study, displayed in Fig. 12, indicate excellent agreement between the two independent models. This provides some confidence as to the accuracy of this technique. Moreover, this comparison shows that the strongly accelerated wakefields are produced by combination of the electromagnetic reaction of the acceleration structures walls with the beam-generated field.

7. Conclusions

RF-photoinjectors provide a source of high-brightness beams which fulfill the requirement of many light source and accelerator applications from X-ray free electron lasers to electron cooling for
high-energy colliders [18,19]. One particularly important effect that dilutes the beam quality is the wakefield induced by the electron beam itself. Routinely in accelerator physics, the wakefield is often introduced as a relativistic self-field in which the beam is assumed to be moving at a constant velocity. In linacs for example [2,3], in the ultra-relativistic regime, this uniform velocity assumption is well-justified. However, in photocathodes, this assumption is unwarranted as the beam is in the low-energy regime in which it is rapidly accelerated up from low velocities. Furthermore, the charges image on the cathode must be taken into account explicitly in this regime.

Based on Lienard–Wiechert retarded potentials and the potential due to charges image on the cathode, rigorous analytical expressions for the self-field and the field generated by charges image on the cathode have been obtained. In these expressions, the time-dependent component of the beam’s velocity has been explicitly taken into account. Simulations have been performed for parameters appropriate to the ELSA photocathode. These simulations show that at the center of the cathode (r = 0, z = 0) the self-field of the beam and the field driven by the charges image on the cathode are equal. The self-field of the beam is dominated by the field due to the charges image on the cathode as one moves from the tail to the head of the beam. Far from the cathode, the self-field becomes the dominant one.

It is hoped that this analysis will provide guidance to those studying means to enhance the brightness of electron beam sources.

References