EIGENVALUE LOCALIZATION FOR COMPLEX MATRICES

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Abstract. Let $A$ be an $n \times n$ complex matrix with $n \geq 3$. It is shown that at least $n - 2$ of the eigenvalues of $A$ lie in the disk

$$|z - \frac{\text{tr} A}{n}| \leq \left( \frac{n - 1}{n} \left( \frac{\|A\|^2}{2} - \frac{\|A - AA^*\|^2}{2} - \frac{\text{spd}(A)}{2} \right) \right)^{1/2},$$

where $\|A\|$, $\text{tr} A$, and $\text{spd}(A)$ denote the Frobenius norm, the trace, and the spread of $A$, respectively. In particular, if $A = [a_{ij}]$ is normal, then at least $n - 2$ of the eigenvalues of $A$ lie in the disk

$$\left| z - \frac{\text{tr} A}{n} \right| \leq \sqrt{\frac{n - 1}{n} \left( \frac{\|A\|^2}{2} - \frac{\|A - AA^*\|^2}{2} - \frac{3}{2} \max_{i,j=1,\ldots,n} \left( \sum_{k=1 \atop k \neq i}^{n} |a_{ki}|^2 + \sum_{k=1 \atop k \neq j}^{n} |a_{kj}|^2 + \frac{|a_{ii} - a_{jj}|^2}{2} \right) \right)}.$$ 

Moreover, the constant $\frac{3}{2}$ can be replaced by 4 if the matrix $A$ is Hermitian.

Key words. Eigenvalue, Localization, Frobenius norm, Trace, Spread, Normal matrix.

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1. Introduction. Let $M_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices. For a matrix $A \in M_n(\mathbb{C})$, let $\lambda_j(A)$, $j = 1, \ldots, n$, be the eigenvalues of $A$ repeated according to multiplicity, and let the symbols $\|A\|_2$ and $\text{tr} A$ denote the Frobenius norm and the trace of $A$, respectively. We have to keep in mind that the Frobenius norm is unitarily invariant, that is, $\|UAV\|_2 = \|A\|_2$ for all unitary matrices $U, V$ in $M_n(\mathbb{C})$ and $\text{tr} A = \sum_{j=1}^{n} \lambda_j(A)$.

An estimate for the eigenvalues of matrices \cite{14} says that if $A$ is an $n \times n$ real