Singular value inequalities for the arithmetic, geometric and Heinz means of matrices

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Let $A$ and $B$ be positive definite $n \times n$ matrices such that $A \geq B$. Among other results, it is shown that

$$s_j\left(\frac{A + B}{2} - A_{1/2} B\right) \leq \frac{1}{8} s_j\left(B^{-1/2} (A - B)^2 B^{-1/2}\right)$$

and

$$s_j\left(\frac{A + B}{2} - A_{1/2} B\right) \geq \frac{1}{8} s_j\left(A^{-1/2} (A - B)^2 A^{-1/2}\right)$$

for $j = 1, 2, \ldots, n$, where $A_{1/2} B$ is the geometric mean of $A$ and $B$ and $s_j(X)$, $j = 1, 2, \ldots, n$, are the singular values of an $n \times n$ matrix $X$.

Keywords: arithmetic mean; geometric mean; Heinz mean; positive definite matrix; positive semidefinite matrix

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1. Introduction

Let $M_n(\mathbb{C})$ denotes the algebra of all $n \times n$ complex matrices. For Hermitian matrices $A, B \in M_n(\mathbb{C})$, if $A - B$ is positive semidefinite, then we write $A \succeq B$. Likewise, we say that $A$ is positive definite if $A$ is an invertible positive semidefinite matrix.

The singular values of a matrix $A \in M_n(\mathbb{C})$, denoted by $s_j(A)$, $j = 1, 2, \ldots, n$, are the eigenvalues of the positive semidefinite matrix $|A| = (A^* A)^{1/2}$ arranged in decreasing order and repeated according to multiplicity. Weyl’s monotonicity principle [2] for positive semidefinite matrices says that if $A, B \in M_n(\mathbb{C})$ are positive semidefinite such that $A \succeq B$, then $s_j(A) \geq s_j(B)$ for $j = 1, 2, \ldots, n$.

For two matrices $A, B \in M_n(\mathbb{C})$ such that $A$ is positive definite and $B$ is positive semidefinite, define the matrix $A_{1/2} B$ to be

$$A_{1/2} B = A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}$$

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