Telescoping Decomposition Method for Solving First Order Nonlinear Differential Equations

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Abstract **The Telescoping Decomposition Method (TDM) is a new iterative method to obtain numerical and analytical solutions for first order nonlinear differential equations. The method is a modified form of the well-known Adomian Decomposition Method (ADM) where the Adomian polynomials have not to be calculating. The (TDM) is easier to apply and offers better accuracy than the (ADM). Also, it can be applied to other systems where the (ADM) does not work. The (TDM) is proved to be convergent to the exact solution while it is not the case in the (ADM). The idea of the (TDM) can be developed to deal with various types of functional equations, as well.**

Keywords: IVP's; (ADM)
MSC: 34A12; 49M27

1 Introduction

Differential equations appear in various applications in the physical sciences and engineering. They also, have been used to study solutions of partial differential equations, since most techniques reduce the partial differential equation into a differential or a system of differential equations. Most of differential equations coming from real life applications are nonlinear and the seek of analytical solutions is a difficult task. Therefore, numerical methods have been introduced. The Runge-Kutta methods, linear multistep methods and Galerkin method can be used to integrate differential equations numerically and obtain accurate solutions [7, 8]. Recently, certain methods that produce accurate and analytical solutions have been introduced. Such as the tanh method [11], the variational iteration method [9] and the Adomian decomposition method (ADM). The (ADM) method was first introduced by Adomian [1, 2], and it has been used to integrate various systems of functional equations [3, 4]. A main part of the (ADM) is finding the Adomian polynomials. Several Authors have discussed this issue and obtained different approaches for calculating the Adomian polynomials [6, 12]. However, the most popular one is the formula obtained in [1, 3]

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} f\left(\sum_{i=0}^{\infty} u_i \lambda^i\right)|_{\lambda=0},$$

where A_n denotes the Adomian polynomial of degree n, $u = \sum_{i=0}^{\infty} u_i$ is the exact solution of the problem and $f(u)$ is the nonlinear term in the equation. It is worth noting that calculating the Admian polynomials is difficult for large n and the above formula can not be applied if f is a function of more than one variable, such as $f = f(u, u')$. Also, the (ADM) is shown to be divergent for certain problems [10].

In this paper we introduce the Telescoping decomposition method (TDM) for solving first order nonlinear initial value problems. We will use the idea of the Adomain method but avoid calculating the Adomian polynomials. In section 2, we present the expansion procedure of the (TDM) and then show the convergence of the (TDM) in Section 3. We present some numerical results and comparison with the (ADM) in Section 4, and finally we write some concluding remarks in Sections 5.

2 The Expansion Procedure

We consider the initial value problem

$$u_t = f(t, u, u_t), \quad t \in \Omega \tag{1}$$
$$u(0) = u_0, \tag{2}$$

where $\Omega = [0, T]$ is compact subset of $\mathbf{R}$. By integrating the above equation we have

$$u(t) = u(0) + \int_0^t f(\tau, u(\tau), u_\tau(\tau))d\tau. \tag{3}$$

We consider a solution of the form $u(t) = \sum_{n=0}^{\infty} u^n(t)$, where $u^n(t)$ has to be determined sequentially upon the following algorithm:

$$\begin{aligned} u^0 &= u_0, \qquad (4)\\ u^1 &= \int_0^t f\left(\tau, u^0(\tau), u^0_\tau(\tau)\right) d\tau,\\ u^2 &= \int_0^t f\left(\tau, \sum_{k=0}^{1} u^k(\tau), \sum_{k=0}^{1} u^k_\tau(\tau)\right) d\tau - \end{aligned}$$

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$$\begin{aligned}
& \int_0^t f\left(\tau, u^0(\tau), u^0_\tau(\tau)\right) d\tau, \\
u^3 &= \int_0^t f\left(\tau, \sum_{k=0}^{2} u^k(\tau), \sum_{k=0}^{2} u^k_\tau(\tau)\right) d\tau - \\
& \int_0^t f\left(\tau, \sum_{k=0}^{1} u^k(\tau), \sum_{k=0}^{1} u^k_\tau(\tau)\right) d\tau, \\
u^4 &= \int_0^t f\left(\tau, \sum_{k=0}^{3} u^k(\tau), \sum_{k=0}^{3} u^k_\tau(\tau)\right) d\tau - \\
& \int_0^t f\left(\tau, \sum_{k=0}^{2} u^k(\tau), \sum_{k=0}^{2} u^k_\tau(\tau)\right) d\tau, \\
& . \\
& . \\
& . \\
u^n &= \int_0^t f\left(\tau, \sum_{k=0}^{n-1} u^k(\tau), \sum_{k=0}^{n-1} u^k_\tau(\tau)\right) d\tau - \\
& \int_0^t f\left(\tau, \sum_{k=0}^{n-2} u^k(\tau), \sum_{k=0}^{n-2} u^k_\tau(\tau)\right) d\tau. \qquad (5)
\end{aligned}$$

Adding equations between (4) and (5) we have

$$\sum_{k=0}^{n} u^k(t) = u_0 + \int_0^t f\left(\tau, \sum_{k=0}^{n-1} u^k(\tau), \sum_{k=0}^{n-1} u^k_\tau(\tau)\right) d\tau, \quad n \geq 1. \qquad (6)$$

We remind here that the choice of u^0 in (4) is not unique, we can chose it to be any function of t and this depends on the problem as we will see in Section 4. Also, if $f(t, u, u_t) = u$ the simple linear case then the (ADM) and (TDM) will coincide and give the exact solution of the problem.

3 Convergence analysis

In this section we prove that $\sum_{k=0}^{\infty} u^k$ converges uniformly to the exact solution of (1-2). We have the following lemmas before writing the main result.

Lemma 1 *Consider the sequence of functions $u^n(t)$, $n \geq 0$, as defined in (4-5). If f is differentiable on Ω and $\frac{\partial f}{\partial u}$ is continuous on Ω, then the infinite series $\sum_0^{\infty} u^n$ converges uniformly on Ω.*

Proof 1 *As f and $\frac{\partial f}{\partial u}$ are continuous on Ω, then there exist positive real numbers M and L such that $\|f(\tau, u, u_\tau)\| \leq M$ and $\|\frac{\partial f}{\partial u}\| \leq L$. First we shall use mathematical induction to prove the inequality*

$$\|u^n\| \leq \frac{L^{n-1} M t^n}{n!}, \quad \forall n \geq 1. \qquad (7)$$

The result is true for $n = 1$, since $\|u^1\| = \|\int_0^t f(t, u^0, u^0_\tau) d\tau\| \leq Mt$.

Suppose that for $k \geq 2$, $\|u^k\| \leq \frac{L^{k-1} M t^k}{k!}$, we shall show that the inequality holds for $k+1$. Recall

$$u^{k+1} = \int_0^t [\ f\left(\tau, \sum_{i=0}^{k} u^i(\tau), \sum_{i=0}^{k} u^i_\tau(\tau)\right) d\tau$$

$$- \int_0^t f\left(\tau, \sum_{i=0}^{k-1} u^i(\tau), \sum_{i=0}^{k-1} u^i_\tau(\tau)\right)]d\tau.$$

By applying the Mean Value Theorem on the second variable of f, there exists ζ_k such that

$$f\left(\tau, \sum_{i=0}^{k} u^i(\tau), \sum_{i=0}^{k} u^i_\tau(\tau)\right) - f\left(\tau, \sum_{i=0}^{k-1} u^i(\tau), \sum_{i=0}^{k-1} u^i_\tau(\tau)\right)$$

$$= \frac{\partial f}{\partial u}(\zeta_k)\ u^k(\tau),$$

where ζ_k lies on the line segment $(1-\lambda)\sum_{i=0}^{k} u^i(\tau) + \lambda \sum_{i=0}^{k-1} u^i(\tau)$, for $\lambda \in (0,1)$. Then we have

$$\|u^{k+1}\| \leq \int_0^t \|\frac{\partial f}{\partial u}(\zeta_k)\|\ \|u^k\| d\tau,$$

therefore by the induction hypothesis, we get

$$\begin{aligned}
\|u^{k+1}\| &\leq \frac{L^{k-1} M}{k!} \int_0^t \|\frac{\partial f}{\partial u}(\zeta_k)\| t^k d\tau \\
&\leq \frac{L^k M t^{k+1}}{(k+1)!},
\end{aligned}$$

and the inequality is proved.

The supremum norm of u_n on Ω satisfies

$$\|u^n\|_\Omega = \sup\{\|u^n(t)\|;\ t \in \Omega\} \leq \frac{L^{n-1} M T^0}{n!}.$$

Now as $\sum_{n=0}^{\infty} \frac{L^{n-1}}{n!}$ is convergent series of positive real numbers, then by using the Weierstrass M-test ([5],p.317), we have $\sum_{n=0}^{\infty} u^n$ is uniformly convergent on Ω, hence the lemma has been checked.

Lemma 2 *Consider the sequence of functions $u^n(t)$, $n \geq 0$, as defined in (4-5). If f is differentiable on Ω and $\frac{\partial f}{\partial u}$ is continuous on Ω, then the infinite series $\sum_0^{\infty} u^n_t$ converges on Ω.*

Proof 2 *As f and $\frac{\partial f}{\partial u}$ are continuous on Ω, then there exist positive real numbers M and L such that $\|f(\tau, u, u_\tau)\| \leq M$ and $\|\frac{\partial f}{\partial u}\| \leq L$. Therefore $\|u^1_\tau\| = \|f(t, u_0, 0)\| \leq M$. Now recall that for all $n \geq 2$,*

$$u^n_\tau = f(\tau, \sum_{i=0}^{n-1} u^i(\tau), \sum_{i=0}^{n-1} u^i_\tau(\tau)) - f(\tau, \sum_{i=0}^{n-2} u^i(\tau), \sum_{i=0}^{n-2} u^i_\tau(\tau)).$$

By applying the Mean value theorem on the second variable of f, there exists ζ_{n-1} such that

$$f(\tau, \sum_{i=0}^{n-1} u^i(\tau), \sum_{i=0}^{n-1} u^i_\tau(\tau)) - f(\tau, \sum_{i=0}^{n-2} u^i(\tau), \sum_{i=0}^{n-2} u^i_\tau(\tau))$$
$$= \frac{\partial f}{\partial u}(\zeta_{n-1})\ u^{n-1}(\tau),$$

where ζ_{n-1} lies on the line segment $(1-\lambda)\sum_{i=0}^{n-1} u^i(t) + \lambda \sum_{i=0}^{n-2} u^i(t)$, for some $\lambda \in [0,1]$. Then by using inequality (7) we have

$$\begin{aligned} \|u^n_\tau(t)\| &= \|\frac{\partial f}{\partial u}(\zeta_{n-1})\|\ \|u^{n-1}(t)\| \\ &\le L\left(\frac{L^{n-2}Mt^{n-1}}{(n-1)!}\right) \\ &= \frac{L^{n-1}Mt^{n-1}}{(n-1)!} \end{aligned}$$

Hence we proved the validity of the argument for all $n \ge 1$. Therefore,

$$\|u^n_t\|_\Omega = \sup\{\|u^n_t(t)\|;\ t \in \Omega\} \le \frac{L^{n-1}M\beta}{(n-1)!},$$

for some positive real number β. As $\sum_{n=0}^{\infty} \frac{L^{n-1}}{n!}$ is convergent series of positive real numbers, then by the Weierstrass M-test, the lemma has been proved.

Theorem 1 *Consider the initial value problem (1-2), and the sequence of functions $u^n(t)$, as defined in (4-5). If f is differentiable on Ω and $\frac{\partial f}{\partial u}$ is continuous on Ω, then $\sum_0^\infty u^n$ converges uniformly to the exact solution of the initial value problem.*

Proof 3 *Recall that for every positive integer n,*

$$\sum_{k=0}^{n} u^k = u_0 + \int_0^t f(\tau, \sum_{k=0}^{n-1} u^k, \sum_{k=0}^{n-1} u^k_\tau)d\tau.$$

Let

$$s_n = \sum_{k=0}^{n-1} u^k, \qquad \tilde{s_n} = \sum_{k=0}^{n-1} u^k_\tau$$

and define the sequence of functions g_n by $g_n := f(\tau, s_n, \tilde{s_n})$.

By Lemma (1), we have that s_n is uniformly convergent on Ω to some function V, and by Lemma (2), we get $\tilde{s_n}$ converges uniformly on Ω to V' ([5],p.317).

Moreover, as f is uniformly continuous on Ω, then the sequence g_n converges uniformly to $f(\tau, V, V')$. Therefore,

$$\begin{aligned} \sum_{k=0}^{\infty} u^k &= u_0 + \lim_{n\to\infty}\int_0^t g_n d\tau \\ &= u_0 + \int_0^t \lim_{n\to\infty} g_n d\tau \\ &= u_0 + \int_0^t f(\tau, \lim_{n\to\infty} s_n, \lim_{n\to\infty} \tilde{s_n})d\tau \\ &= u_0 + \int_0^t f(\tau, V(\tau), V'(\tau))d\tau \end{aligned}$$

and hence the result is obtained.

4 Numerical Results

In the following we present various applications to illustrate the validity and effectiveness of the (TDM). We denote $u_T = \sum_{k=0}^{n} u^k$ and $u_A = \sum_{k=0}^{n} u_k$ the approximate solutions obtained by the (TDM) and (ADM), respectively.

Example 1 *Consider the initial value problem*

$$\begin{aligned} u' &= 2 + 2t + 2t^2 + 2t^3 + t^4 - (1+t^2)u^2 \\ u(0) &= 1. \end{aligned}$$

The problem has been discussed in [10] and the infinite series solution obtained by the (ADM) is proved to be divergent. Since $f(t,u)$ is differentiable and $\frac{\partial f}{\partial u} = -2(1+t^2)u$ is continuous, then the infinite series solution obtained by the (TDM) converges to the exact solution $u(t) = 1+t$. Integrating the equation with respect to t we have

$$u(t) = 1 + 2t + t^2 + \frac{2}{3}t^3 + \frac{1}{2}t^4 + \frac{1}{5}t^5 - \int_0^t (1+\tau^2)u^2(\tau)d\tau.$$

We start with $u^0 = 1 + 2t + t^2 + \frac{2}{3}t^3 + \frac{1}{2}t^4 + \frac{1}{5}t^5$ and the approximate solution $u_T = \sum_{k=0}^{n} u^k$ is obtained by applying the (TDM) algorithm. Figures 1 and 2 show the exact and approximate solutions of the problem obtained by the (TDM) for n=3,4,5 and 6 with $t \in [0, .5]$. Figure 1 show that the approximate solutions u_T are close enough to the exact solution u for $t \in [0\ .25]$. While they are little bit far from the exact solution in [.25 .5]. However, for $n = 4$ and 5 the approximate solutions are very close to the exact one in [0 .5] as shown in Figure 2. That is, we have to increase the number of terms n in order to achieve accurate approximate solution as the time increases.

Example 2 *Consider the initial value problem*

$$\begin{aligned} u' &= 1 + u^2 \\ u(0) &= 0. \end{aligned}$$

The Taylor series expansion of the exact solution

$$u(t) = \tan(t) = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \frac{62t^9}{2835} + \cdots$$

We have $u(t) = u(0)+t+\int_0^t u^2(\tau)d\tau$, and since $u(0) = 0$, we start with $u^0 = t$. Applying the (TDM) we have

$$\begin{aligned} u^1 &= \frac{1}{3}t^3, \\ u^2 &= \frac{2}{15}t^5 + \frac{1}{63}t^7, \\ u^3 &= \frac{4}{105}t^7 + \frac{38}{2835}t^9 + \frac{134}{51975}t^{11} + \frac{4}{12285}t^{13} \\ &\quad + \frac{1}{59535}t^{15}, \\ u^4 &= \frac{8}{945}t^9 + \frac{148}{31185}t^{11} + \frac{11344}{6081075}t^{13} + \frac{366292}{638512875}t^{15} \\ &\quad + \frac{1522814}{10854718875}t^{17} + \cdots + \frac{1}{10987690297}t^{31}. \end{aligned}$$

Hence,

$$u^0 + u^1 + u^2 + u^3 + u^4 + \cdots = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \frac{62t^9}{2835} + O(t^{11}),$$

which coincides with the Taylor series of the exact solution. That is, the exact solution of the problem is obtained by the (TDM).

Example 3 *Consider the initial value problem*

$$\begin{aligned} u' &= 1 + 2u - u^2 \\ u(0) &= 0. \end{aligned}$$

The exact solution of the problem $u(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + 2\log(\frac{\sqrt{2}-1}{\sqrt{2}+1})\right)$. Figure 3 and 4 present the exact and approximate solutions u_A and u_T of the problem obtained by the (ADM) and (TDM), respectively, for $n = 2, 3, 4$ and $t \in [0 \;\; 1.5]$. One can see that the (ADM) and the (TDM) produce solutions which converge to the exact solution of the problem but the (TDM) converges faster. To illustrate this conclusion, we compute the errors e_A and e_T between the exact and approximate solutions obtained by the (ADM) and (TDM), respectively. We define

$$e_A = ||u - u_A||_2 = \sqrt{\int_0^T (u - u_A)^2 dt},$$

and in a similar manner we define e_T. Tables 1-5 present the errors e_A and e_T for different values of n and T. One can see that both errors increase as T increases. For $T = 1.5$ the error e_A is not necessarily decreasing with n, which indicates the (ADM) produces unstable solution. While, the error e_T is decreasing with n in all cases of T. Also, we have $e_T < e_A$ except for the case when $T = 0.5$ and $n = 2$ as indicated in Table 1. The above discussion demonstrates that the (TDM) is more efficient and converges faster than the (ADM) and we expect it to have the same feature for other problems, as well.

Table 1: The errors between the exact and approximate solutions obtained by the (ADM) and (TDM) for n=2.

	e_A	e_T
$0.0 \le t \le 0.5$	0.0009246	0.0018798
$0.0 \le t \le 1.0$	0.0336764	0.0048257
$0.0 \le t \le 1.5$	0.1727480	0.0557244

Table 2: The errors between the exact and approximate solutions obtained by the (ADM) and (TDM) for n=3.

	e_A	e_T
$0.0 \le t \le 0.5$	0.0002836	0.0002096
$0.0 \le t \le 1.0$	0.0135943	0.0005716
$0.0 \le t \le 1.5$	0.0505807	0.0115319

Table 3: The errors between the exact and approximate solutions obtained by the (ADM) and (TDM) for n=4.

	e_A	e_T
$0.0 \le t \le 0.5$	0.0000854	0.0000204
$0.0 \le t \le 1.0$	0.0027531	0.0000555
$0.0 \le t \le 1.5$	0.0753189	0.0072910

5 Concluding Remarks

We have presented and analyzed the Telescoping Decomposition Method (TDM) for solving nonlinear first order initial value problem of the form

$$u_t = f(t, u, u_t), \quad u(0) = u_0, \; t \in \Omega,$$

where Ω is a compact subset of R. We proved that the (TDM) converges uniformly to the exact solution of the problem provided that f is differentiable and $\frac{\partial f}{\partial u}$ is continuous on Ω. We have applied the (TDM) for a variety of applications and obtained an accurate solutions as well as, exact solutions for certain problems.

The (TDM) has several advantages over the Adomian Decomposition Method (ADM): no need to calculate the Adomian polynomials and replace them by a simple formula, the (TDM) always converges to the exact solution while the (ADM) is not, and it converges faster to the exact solution. Also, there is no general formula for calculating the Adomian polynomials with nonlinear term $f(t, u, u_t)$.

The idea of the (TDM) can easily be modified to obtain solutions of higher order differential equations but proving the convergence of the method is not an easy task. Also, it can be applied for various systems of partial and integral equations, as well. However, we leave these issues for a future work.

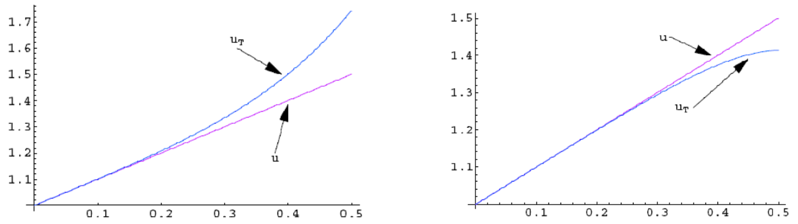

Figure 1: The exact solution u and the approximate solution u_T obtained by the (TDM) for $n = 3$ (left) and $n = 4$ (right) and $t \in [0, .5]$.

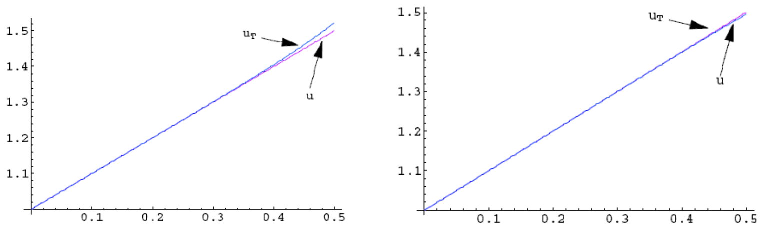

Figure 2: The exact solution u and the approximate solution u_T obtained by the (TDM) for $n = 5$ (left) and $n = 6$ (right) and $t \in [0, .5]$.

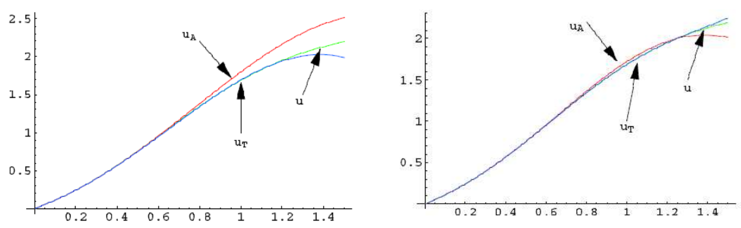

Figure 3: The exact solution u and the approximate solutions u_A and u_T for $n = 2$ (left) and $n = 3$ (right) and $t \in [0, 1.5]$.

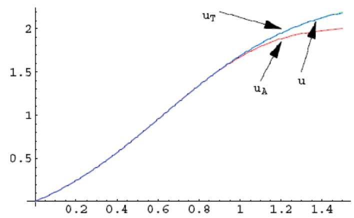

Figure 4: The exact solution u and the approximate solutions u_A and u_T for $n = 4$ and $t \in [0, 1.5]$.

Table 4: The errors between the exact and approximate solutions obtained by the (ADM) and (TDM) for n=5.

	ADM	TDM
$0.0 \le t \le 0.5$	$7.59768 * 10^{-6}$	$1.75221 * 10^{-6}$
$0.0 \le t \le 1.0$	0.0029373	$5.04844 * 10^{-6}$
$0.0 \le t \le 1.5$	0.0169083	0.0001582

Table 5: The errors between the exact and approximate solutions obtained by the (ADM) and (TDM) for n=6.

	ADM	TDM
$0.0 \le t \le 0.5$	$1.76954 * 10^{-6}$	$1.34047 * 10^{-7}$
$0.0 \le t \le 1.0$	0.0004265	$4.07328 * 10^{-7}$
$0.0 \le t \le 1.5$	0.0353069	0.0000394

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