Gaseous Slip Flow in Three-Dimensional Uniform Rectangular Microchannel

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Abstract. This paper analyzes compressible gaseous slip flow through a three-dimensional straight uniform rectangular microchannel, and reports a set of asymptotic solutions. By comparing the magnitudes of different forces in the compressible gas flow, we obtain a proper criteria to estimate the Reynolds and Mach numbers at the channel exit. We select two sets of Mach and Reynolds numbers and obtain asymptotic analytical solutions of velocities and pressure distributions; first order velocity slip and non-slip boundary conditions are examined. The analytical results of pressure and velocities are compared with numerical simulation results of direct simulation Monte Carlo method and the results that are available in the literature.

Keywords: three-dimensional rectangular microchannel, asymptotic solutions, gaseous slip flow
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INTRODUCTION

Because of the wide applications of microchannels in micro-electro-mechanical systems(MEMS), it is desired to investigate physical gaseous flows inside these channels to effectively design and optimize the channels.

In the literature, there is much work on this subject, and here for the literature review we concentrate on gaseous slip flow inside a straight microchannel. Arkilic et al. [1] conducted an analytical and experimental investigation on gaseous flow through long microchannels with slight rarefaction; they derived a set of two-dimensional theoretical formula for the nonlinear pressure, velocity profiles and mass flow rate of gaseous flow in microchannels. Chen [2, 3] studied numerically gas flow in two-dimensional and three-dimensional microchannels. Cai et al. [4] presented a complete set of first-order analytical solutions for compressible gas flow inside a two-dimensional or axis-symmetric uniform microchannel with a relaxation of the isothermal assumption. Jang and Wereley [5] reported that the pressure inside a rectangular microchannel is nonlinear. Hsieh et al.[6] conducted an experimental and theoretical study of low Reynolds number compressible gas flow in a microchannel; his experimental results were found in good agreement with those predicted by analytical solutions. Jain and Lin [7] presented numerical results for three-dimensional nitrogen gas flows in microchannels with slip and non-slip boundary conditions. Ebert and Sparrow [8] performed a study for the flow in rectangular and annular ducts for rarefied gas flow, and found that the effect of slip is to flatten the velocity distribution relative to that for continuum flow and the axial pressure gradient is diminished under the slip conditions. Guo et al.[9] conducted an experiment to study the flows in two- and three-dimensional microchannels. Qin et al. [10] studied theoretically a two-dimensional steady subsonic gas flow either in a circular micropipe or in a planar microchannel driven by pressure within the slip flow regime, high-order boundary conditions of velocity slip and temperature jump are adopted at the wall.

The objectives of the present work are to obtain a set of asymptotic solutions of velocities and pressure for the compressible gas flow inside a long, uniform, three-dimensional rectangular microchannel using the Navier-Stokes equations and different boundary conditions.

PHYSICAL PROBLEM DESCRIPTION AND GOVERNING EQUATIONS

The microchannel under consideration has three-dimensions with a height of H, a length of L and a width of W, the origin is set at the center of the channel inlet and we choose the flow direction as the x-axis. We consider two cases of boundary conditions [2]: the non-slip boundary conditions case, $u_w = v_w = w_w = 0$, the first order slip boundary conditions case, $u_w = \theta_u \lambda(x)(\frac{\partial u}{\partial n})_w$ at $y = \pm H/2$ or $z = \pm W/2$ and $v_w = w_w = 0$ where $\theta_u = \frac{2-\sigma_u}{\sigma_u}$, σ_u is the momentum accommodation coefficient, $\lambda(x)$ is the local mean free path at specific cross section. We normalize the flow properties with the averaged values at the channel exit: $p_o, \rho_o, T_o, U_o, n_o$, and x, y, z coordinates with the channel length, height and width. We can assume that the viscosity coefficient μ and the heat conductivity k are constant, then the nondimensional governing equations are:

$$\varepsilon \frac{\partial(\rho u)}{\partial x} + A_1 \frac{\partial(\rho v)}{\partial y} + A_2 \frac{\partial(\rho w)}{\partial z} = 0 \tag{1}$$

$$\varepsilon \rho u \frac{\partial u}{\partial x} + A_1 \rho v \frac{\partial u}{\partial y} + A_2 \rho w \frac{\partial u}{\partial z} = -\frac{\varepsilon}{\gamma M^2}\frac{\partial p}{\partial x} + \frac{1}{Re}\left[\varepsilon^2 \frac{\partial^2 u}{\partial x^2} + A_1^2 \frac{\partial^2 u}{\partial y^2} + A_2^2 \frac{\partial^2 u}{\partial z^2} + \frac{1}{3}\left(\varepsilon^2 \frac{\partial^2 u}{\partial x^2} + \varepsilon A_1 \frac{\partial^2 v}{\partial x \partial y} + \varepsilon A_2 \frac{\partial^2 w}{\partial x \partial z}\right)\right] \tag{2}$$

$$\varepsilon \rho u \frac{\partial v}{\partial x} + A_1 \rho v \frac{\partial v}{\partial y} + A_2 \rho w \frac{\partial v}{\partial z} = -\frac{A_1}{\gamma M^2}\frac{\partial p}{\partial y} + \frac{1}{Re}\left[\varepsilon^2 \frac{\partial^2 v}{\partial x^2} + A_1^2 \frac{\partial^2 v}{\partial y^2} + A_2^2 \frac{\partial^2 v}{\partial z^2} + \frac{1}{3}\left(\varepsilon A_1 \frac{\partial^2 u}{\partial x \partial y} + A_1^2 \frac{\partial^2 v}{\partial y^2} + A_1 A_2 \frac{\partial^2 w}{\partial y \partial z}\right)\right] \tag{3}$$

$$\varepsilon \rho u \frac{\partial w}{\partial x} + A_1 \rho v \frac{\partial w}{\partial y} + A_2 \rho w \frac{\partial w}{\partial z} = -\frac{A_2}{\gamma M^2}\frac{\partial p}{\partial z} + \frac{1}{Re}\left[\varepsilon^2 \frac{\partial^2 w}{\partial x^2} + A_1^2 \frac{\partial^2 w}{\partial y^2} + A_2^2 \frac{\partial^2 w}{\partial z^2} + \frac{1}{3}\left(\varepsilon A_2 \frac{\partial^2 u}{\partial x \partial z} + A_1 A_2 \frac{\partial^2 v}{\partial y \partial z} + A_2^2 \frac{\partial^2 w}{\partial z^2}\right)\right] \tag{4}$$

$$p = \rho T = nT \tag{5}$$

where A_1,A_2 are ratios of hydrodynamic diameter to duct height and duct width, $A_1 = D_h/H$ and $A_2 = D_h/W$, $\varepsilon = D_h/L$ and $D_h = 2WH/(W+H)$. The non-dimensional wall boundary conditions at $y = \pm 1/2$ or $z = \pm 1/2$ are: $u_w = v_w = w_w = 0$, for the non slip velocity boundary condition; $u_w = \theta_u Kn(x) F(\frac{\partial u}{\partial n})_w$, $v_w = w_w = 0$ for the first order slip velocity boundary conditions. where $Kn(x) = Kn_o/\rho(x)$ and $F = A_1$ or $F = A_2$ for the y or z direction correspondingly.

ASYMPTOTIC SOLUTIONS

A proper order estimation is needed to simplify the problem. We apply the X-momentum equation for the whole channel, and obtain the following equation regarding outlet Mach and Reynolds numbers:

$$O[(P-1)\varepsilon M^{-2}] - O[\gamma \varepsilon] - O[\gamma/Re] \sim 0 \tag{6}$$

where $P = p_i/p_o$ is the pressure ratio.

We are interested in low speed gas flows, and we choose the following two cases: $M \sim O(\varepsilon)$, $Re \sim O(\varepsilon)$ and $M \sim O(\varepsilon^{1/2})$, $Re \sim O(1)$ for our analysis. Because u is symmetric about the centerline, we can consider only the right-top quarter of the cross section to simplify the problem where the boundary conditions along this line is $\frac{\partial u}{\partial n} = 0$. We define the same formats for the non-dimensional quantities as in [4].

$$\begin{aligned} &u = u_1 + \varepsilon u_2 + \varepsilon^2 u_3 + \dots, \quad v = \varepsilon v_2 + \varepsilon^2 v_3 + \dots, \quad w = \varepsilon w_2 + \varepsilon^2 w_3 + \dots, \quad p = p_1 + \varepsilon p_2 + \dots, \\ &\rho = \rho_1 + \varepsilon \rho_2 + \varepsilon^2 \rho_3 + \dots, \quad n = n_1 + \varepsilon n_2 + \varepsilon^2 n_3 + \dots \end{aligned} \tag{7}$$

The solutions of u_1, v_2, and w_2 can be obtained by considering the leading terms in the x, y and z-momentum equations. From the y- and z- momentum Eqns, the leading term is $O(\frac{1}{\varepsilon^2})$ and it provides us a relation that $\frac{\partial p}{\partial y} = 0$, $\frac{\partial p}{\partial z} = 0$, i.e, p is a function of x only. The leading term in the x-momentum Eqn.(2):

$$A_1^2 \frac{\partial^2 u_1}{\partial y^2} + A_2^2 \frac{\partial^2 u_1}{\partial z^2} = \frac{\varepsilon Re}{\gamma M^2}\frac{dp_1}{dx} \tag{8}$$

We denote the right hand side term as $c_1(x)$ hereafter. The asymptotic solution of u_1 at a specific cross section can be derived from Eqn.(8) with constant $c_1(x)$ for both cases of slip and non-slip conditions, and the solutions of $v_2(x,y,z)$ and $w_2(x,y,z)$ can be derived by substituting the expansion formats [4] into Eqn.(1) and using the condition that $p_1 = \rho_1$. By collecting terms of order ε^1, we obtain:

$$A_1 \frac{\partial v_2}{\partial y} + A_2 \frac{\partial w_2}{\partial z} = -\frac{dp_1}{p_1 dx} u_1 - \frac{\partial u_1}{\partial x} \tag{9}$$

where the right hand side term is defined as $S_1(x,y,z)$. From the above Eqn.(9) we can solve for v_2 and w_2 after we obtain the source term that containing u_1 and applying the wall boundary conditions.

Asymptotic Solution with Non-Slip Velocity Boundary Conditions

The solution of $u_1(x,y,z)$ for the non-slip velocity boundary condition case can be obtained from Eqn.(8) with $u_1 = 0$ at the wall boundaries:

$$u_1(x,y,z) = \frac{4c_1(x)}{A_1^2} \sum_{n=1,3,5,..}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{\lambda_n^3} \left(\frac{\cosh(\lambda_n A_4 z)}{\cosh(\frac{1}{2}\lambda_n A_4)} - 1 \right) \cos(\lambda_n y) \tag{10}$$

where $\lambda_n = n\pi$, $A_4 = A_1/A_2$, the above result seems the same as the dimensional format solution in the book by White,[12], however $c_1(x)$ is not a constant here.

Utilizing the condition that mass flux is constant at all cross sections for a steady state, the following relation for the pressure distribution is obtained:

$$p_1(x) = \sqrt{P^2 + (1-P^2)x} \tag{11}$$

Asymptotic Solutions with the First order Velocity Slip Boundary Conditions

The u_1 solution can be derived from Eqn.(8) with the corresponding boundary conditions for $0 \le y \le 1/2$ and $0 \le z \le 1/2$:

$$u_1(x,y,z) = c_1(x) \left[\sum_{n=1,2,..}^{\infty} D_n(x) \cosh[A_4 \beta_n(x) z] \cos[\beta_n(x) y] + \frac{4y^2 - 1 - 4A_1 \theta_u Kn(x)}{8A_1^2} \right] \tag{12}$$

$$D_n(x) = \frac{4\sin(\frac{\beta_n(x)}{2})}{A_1^2 \beta_n^2 [A_1 \beta_n(x) \theta_u Kn(x) \sinh(\frac{1}{2} A_4 \beta_n(x)) + \cosh(\frac{1}{2} A_4 \beta_n(x))]} \frac{1}{\sin\beta_n(x) + \beta_n(x)} \tag{13}$$

The solutions of $v_2(x,y,z)$ and $w_2(x,y,z)$ for the slip case are obtained by solving the following equation:

$$A_1 \frac{\partial v_2}{\partial y} + A_2 \frac{\partial w_2}{\partial z} = S_2(x,y,z) = \left(-\frac{1}{p_1}\frac{dp_1}{dx} - \frac{\partial}{\partial x} \right) u_1 = k(x) u_1(x,y,z) \tag{14}$$

where $u_1(x,y,z)$ is known, $k(x) = \left(-\frac{1}{p_1}\frac{dp}{dx} - \frac{\partial}{\partial x} \right)$ is a mathematical operator. It can be proved that the final expressions for the $v_2(x,y,z)$, $w_2(x,y,z)$ solutions are:

$$v_2(x,y,z) = -\sum_{m=1,2,3,..}^{\infty} \gamma_m \left(E_m(x) \cosh(A_3 \gamma_m z) + Z_{pm}(x,z) \right) \sin(\gamma_m y) \tag{15}$$

$$w_2(x,y,z) = \sum_{m=1,2,3,..}^{\infty} \left(A_3 \gamma_m E_m(x) \sinh(A_3 \gamma_m z) + \frac{\partial Z_{pm}(x,z)}{\partial z} \right) \cos(\gamma_m y) \tag{16}$$

where $\gamma_m = 2m\pi$, $A_3 = \sqrt{A_4}$, $E_m(x)$, Z_{pm} are:

$$E_m(x) = 4k(x) c_1(x) \sum_{n=1,2,...}^{\infty} D_n(x) l_{nm}(x) \frac{A_4 \beta_n(x)}{A_2 A_3 \gamma_m (A_3^2 \gamma_m^2 - A_4^2 \beta_n(x)^2)} \frac{\sinh(\frac{1}{2} A_4 \beta_n(x))}{\sinh(\frac{1}{2} A_3 \gamma_m)} \tag{17}$$

$$Z_{pm}(x,z) = -2k(x) c_1(x) \left[\frac{(-1)^m}{A_1^3 \gamma_m^4} + 2 \sum_{n=1,2,..}^{\infty} D_n l_{nm} \left(\frac{1}{A_2 (A_3^2 \gamma_m^2 - A_4^2 \beta_n^2(x))} \right) \cosh(A_4 \beta_n(x) z) \right] \tag{18}$$

$$l_{nm}(x) = \frac{-\beta_n(x)}{\beta_n^2(x) - \gamma_m^2} \sin(\frac{\beta_n(x)}{2}), \tag{19}$$

The pressure distribution is obtained from the above equation, with the following format:

$$Q_1 \frac{dp_1^2}{dx} - \frac{\theta_u Kn_o}{4A_2} \frac{dp_1}{dx} = Q_2 \tag{20}$$

where Q_2 is a coefficient related to the mass flow rate, and

$$Q_1 = \sum_{n=1,2,3,..}^{\infty} \left[\frac{4\sin^2\frac{\beta_n(x)}{2}}{A_1^2\beta_n^4\left[A_1\beta_n\theta_u Kn(x)+\coth(\frac{1}{2}A_4\beta_n)\right]} \frac{1}{\sin\beta_n+\beta_n} \right] - \frac{1}{48A_1A_2} \tag{21}$$

the solution to Eqn. (20) is:

$$p_1(x) = \frac{\frac{\theta_u Kn_o}{4A_2} - \sqrt{(\frac{\theta_u Kn_o}{4A_2})^2 + 4Q_1[Q_1(1-P^2)-(1-P)\frac{\theta_u Kn_o}{4A_2}]x + 4Q_1^2P^2 - PQ_1\frac{\theta_u Kn_o}{A_2}}}{2Q_1} \tag{22}$$

VALIDATION AND RESULTS

We simulate a finite length uniform microchannel flow with the direct simulation Monte Carlo (DSMC) method. We adopt one benchmark case which is similar to a test case in another paper, [4] by assuming oxygen gas flows through a microchannel with a length of 15 μm, a width and a height of 0.53 μm. The pressure ratio is 2.5, the outlet pressure is 1 atm, the temperatures for the inlet, outlet, and wall are 300 Kelvin.

Figure (1) shows the normalized centerline pressure distributions. It is clear that the analytical results with the slip and no-slip velocity boundary conditions are in good agreement with the DSMC results.

Figure (2) shows the centerline U-velocity distributions. The U-velocity increases along the X-axis for all cases, and the DSMC results are close to the average value of the three sets of analytical results. The slip boundary case has higher U-velocity than the no-slip case, since the slip boundary condition case provides less wall shear stress. We should point out that the analytical centerline U-velocity is very close to that from the round tube cross section case with a tube diameter of 0.53 μm [4] and it is in good agreement with that for the rectangular cross-section in [5]. Figure (3) shows the contours of U-velocity in the Z=0 plane for slip boundary conditions with the DSMC results, where the solid for the DSMC and dashed lines for slip case. As we can see the results have the same trends with large gradients at the channel exit. Figures (4) shows the contours of U-velocity at the middle station, $x/L = 0.5$, for slip velocity boundary conditions case with the DSMC, the results have very close values.

CONCLUSION

We have presented a set of asymptotic solutions for the compressible gas flowfield inside a three-dimensional straight microchannel with a uniform rectangular cross section. By comparing the different forces, we obtained a fundamental relation, Eqn.(6), which links the Mach and Reynolds numbers at the channel exit, the pressure ratio P, and the channel geometry ratio ε. This relation provides a guideline to choose different Reynolds and Mach number orders to simplify the Navier-Stokes Equations. By utilizing the non-slip boundary conditions, the pressure expressions along the short microchannel, which is nonlinear for the rectangular cross section case, are obtained. This is different from the linear pressure gradient assumption adopted in the classical Poiseuille flow. The velocity distribution u_1 is the same as those reported in the literature, but the nonlinear pressure distribution provides non-constant velocity profiles, while the traditional Poiseuille flow provides us constant velocity profiles at all channel sections. For the slip velocity boundary condition case, we provided the full solutions of u_1, v_2, w_2, and p_1 for first order velocity slip boundary conditions. They can serve as the benchmark test cases for further studies. For the pressure distribution, the format is different from those obtained in the literature.

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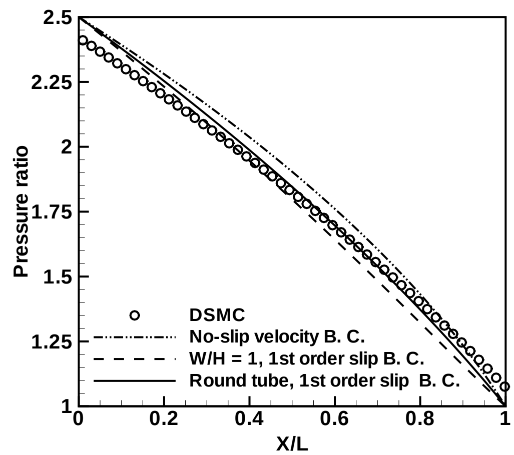

FIGURE 1. Center line pressure along flow direction, W = H = 0.53 μm, L=15 μm, O_2.

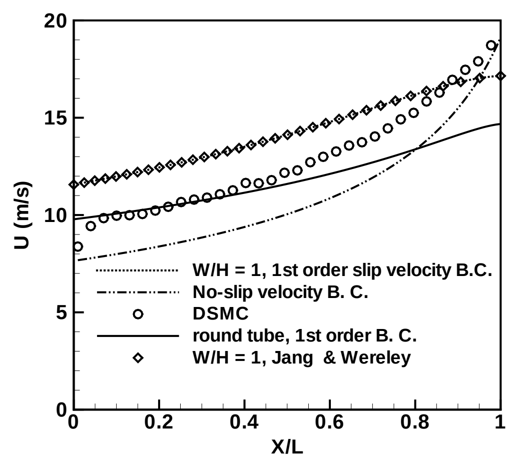

FIGURE 2. Center line U-velocity along flow direction (m/s), W = H = 0.53 μm, L=15 μm, O_2.

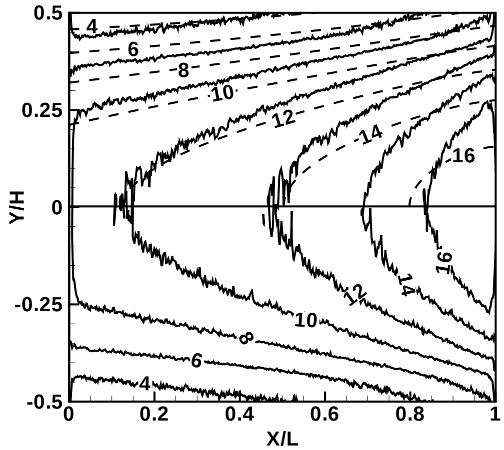

FIGURE 3. U-velocity contours (m/s) in plane $z = 0$, solid: DSMC, dashed: slip velocity B.C.s, W = H= 0.53 μm, L = 15 μm, oxygen.

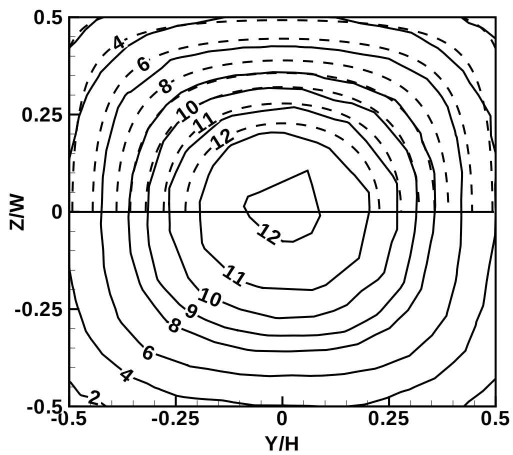

FIGURE 4. U-velocity contours (m/s) at x/L=0.5, solid: DSMC, dashed: slip velocity B.C.s. W = H= 0.53 μm, L = 15 μm, O_2.