FRACTIONAL EULER METHOD AND FINITE DIFFERENCE FORMULA USING CONFORMABLE FRACTIONAL DERIVATIVE

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Abstract- In this paper we use the new definition of fractional derivative called conformable fractional derivative to derive some finite difference formulas and its error terms which are used to solve fractional differential equations and fractional partial differential equations, also to derive fractional Euler method and its error terms which can be applied to solve fractional differential equations. To provide the contribution of our work, some applications on finite difference formulas and Euler Method are given.

Index Terms- Conformable fractional derivative, Finite difference formula, Fractional derivative, Finite difference formula.

I. INTRODUCTION

THE concept of fractional derivative was known by L'Hopital when he asked Leibniz in the year 1695 about the differentiation of order 1/2, in 30 September 1995, Leibniz reply "This is an apparent Paradox from which one day, useful consequences will be drawn".

Since that time many definitions on fractional derivatives are given by many authors: Liouville, Riemann-Liouville, Caputo, Grunwald-Letnikov, Weyl, Marchaud, Chen, Davidson-Essex, Coimbra, Canavati, Jumarie, Riesz, Cossar, Yang, Osler, Erd´elyim Kober and many others definitions. [1]

Most of these definitions do not satisfy the following properties:

1) The known formula of the derivative of the product of two functions: $D_a^\alpha(fg) = fD_a^\alpha(g) + fD_a^\alpha(f)$.
2) The known formula of the derivative of the quotient of two functions: $D_a^\alpha(f/g) = \frac{gD_a^\alpha(f) - fD_a^\alpha(g)}{g^2}$.
3) The chain rule: $D_a^\alpha(fog)(t) = f^\alpha(g(t))g^\alpha(t)$.
4) $D^\alpha(c) = 0$, for any constant c.

Recently, Khalil et.al introduced an interesting definition of fractional derivative called conformable fractional derivative:

$T_\alpha(f(t)) = \lim_{h\to 0} \frac{f(t+ht^{1-\alpha}) - f(t)}{h}$, which satisfied the following properties [2]:

1) Rolle's Theorem.
2) Mean Value Theorem.
3) $T_\alpha(fg) = fT_\alpha(g) + fT_\alpha(f), \alpha \in (0,1]$.
4) $T_\alpha(f/g) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}, \alpha \in (0,1]$.
5) $T_\alpha(fog)(t) = f^\alpha(g(t))g^\alpha(t), \alpha \in (0,1]$.
6) $T_\alpha(c) = 0$, for any constant $c, \alpha \in (0,1]$.
7) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in R, \alpha \in (0,1]$.

II. FINITE DIFFERENCE FORMULAS

The Lagrange interpolating polynomial of the function $f \in C^2[a,b]$ and its error term using two distinct points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ where $x_0, x_1 \in [a,b]$ is given by [ ]:

$$f(x) = \frac{f(x_0)(x-x_1)}{x_0 - x_1} + \frac{f(x_1)(x-x_0)}{x_1 - x_0} + \frac{1}{2}(x-x_0)(x-x_1)f''(\xi(x)), \quad (1)$$

For some ξ between x_0 and x_1.
Let $h = x_1 - x_0$, then Eq.(1) 1 become:

$$f(x) = \frac{f(x_0)(x-x_0-h)}{h} + \frac{f(x_1)(x-x_0)}{h} + \frac{1}{2}(x-x_0)(x-x_0-h)f''(\xi(x)), \quad (2)$$

For $\alpha \in (0,1]$ take T_α to both sides of Eq.(1) gives:

$$T_\alpha(f(x)) = \frac{f(x_0)x^{1-\alpha}}{-h} + \frac{f(x_1)x^{1-\alpha}}{h} + T_\alpha\left(\frac{1}{2}(x-x_0)(x-x_0-h)f''(\xi(x))\right), \quad (3)$$

Substitute $x = x_0$ in the error term in Eq.(3) gives

$$\text{Error term} = \frac{-h}{2} f''(\xi(x)). \quad (4)$$

Thus,

$$T_\alpha(f(x_0)) = {x_0}^{1-\alpha} \frac{f(x_1) - f(x_0)}{h} - \frac{h}{2} f''(\xi(x)) \quad (5)$$

Using the same process, and using three distinct points $(x_0, f(x_0)), (x_1, f(x_1))$ and $(x_2, f(x_2))$, we have the following three-point formulas where $h = x_2 - x_1 = x_1 - x_0$:

5) Forward-difference formula

$$T_\alpha(f(x_0)) = {x_0}^{1-\alpha} \frac{-3f(x_0) + 4f(x_0+h) - f(x_0+2h)}{2h} + \frac{h^2}{3} f'''(\xi(x)) \quad (6)$$

6) Central-difference formula

$$T_\alpha\big(f(x_0+h)\big) = {x_0}^{1-\alpha}\,\frac{-f(x_0)+f(x_0+2h)}{2h} - \frac{h^2}{6}f'''\big(\xi(x)\big) \qquad (7)$$

$$T_\alpha\big(f(x_0+2h)\big) = {x_0}^{1-\alpha}\,\frac{f(x_0)-4f(x_0+h)+3f(x_0+2h)}{2h} + \frac{h^2}{3}f'''\big(\xi(x)\big) \qquad (13)$$

7) Backward-difference formula

III. FRACTIONAL EULER'S METHOD

Consider the following fractional differential equation:

$$y^\alpha(t) = f(t,y), \quad y(a) - y_0, a \le t \le b. \qquad (8)$$

Taylor series of $y(t)$ about $t = t_i$ is given by:

$$y(t) = y(t_i) + y'(t_i)(t-t_i) + \frac{y''(\xi)(t-t_i)^2}{2}, \qquad (9)$$

for some ξ between t and t_i.

Since $y^{(\alpha)}(t) = t^{(1-\alpha)}y'(t)$, we have

$$y(t) = y(t_i) + t^{(\alpha-1)}y^{(\alpha)}(t_i)(t-t_i) + \frac{y''(\xi)(t-t_i)^2}{2}, \qquad (10)$$

If we Divided the interval $[a,b]$ into $n+1$ numbers:

$t_i = a + h(i), i = 0,1,\dots n,$ and assume that

$w_i \approx y_i, i = 0,1,\dots,n,$ (where the notation $y_i = y(t_i)$) then

$$y_{i+1} = y_i + ht_{i+1}^{1-\alpha}f(t_i,y_i) + \frac{y''(\xi)h^2}{2}, \qquad (11)$$

Thus the Fractional Euler method can be written as:

$$w_{i+1} = w_i + ht_{i+1}^{1-\alpha}f(t_i,w_i). \qquad (12)$$

IV. APPLICATIONS

Example 1

Consider the function $f_1(x) = \mathrm{Cos}(\sqrt{x}+3)$.

According to (6), table (1) shows the approximation of $f_1^{(\alpha)}(1.5)$ using different values of h and α.

Table 1
Approximation of $f_1^{(\alpha)}(1.5)$ using different values of h and α according to (6)

α	h	Approximation	Absolute Error
0.1	0.1	0.51949797	7.42784×10^{-5}
0.1	0.01	0.51949797	8.50123×10^{-7}
0.1	0.001	0.51949713	8.62040×10^{-9}
0.1	0.0001	0.51949712	9.02726×10^{-11}
0.5	0.1	0.44178273	6.31577×10^{-5}
0.5	0.01	0.44172029	7.22845×10^{-7}
0.5	0.001	0.44171958	7.32978×10^{-9}
0.5	0.0001	0.44171957	7.67573×10^{-11}
0.9	0.1	0.37564034	5.37019×10^{-5}
0.9	0.01	0.37558726	6.14623×10^{-7}
0.9	0.001	0.37558665	6.23238×10^{-9}
0.9	0.0001	0.37558664	6.52654×10^{-11}

Example 2

Consider the function $f_2(x) = xe^x$.

According to (6), table (2) shows the approximation of $f_2^{(\alpha)}(1.5)$ different values of h and α.

Table 2
Approximation of $f_2^{(\alpha)}(1.5)$ using different values of h and α according to (6)

α	h	Approximation	Absolute Error
0.1	0.1	16.03230819	1.06218×10^{-1}
0.1	0.01	16.13754870	9.77237×10^{-4}
0.1	0.001	16.13851624	9.6920×10^{-6}
0.1	0.0001	16.13852584	9.68555×10^{-8}
0.5	0.1	13.63199911	9.03151×10^{-2}
0.5	0.01	13.72148333	8.30928×10^{-4}
0.5	0.001	13.72230601	8.24094×10^{-6}
0.5	0.0001	13.72231417	8.23546×10^{-8}
0.9	0.1	11.59105710	7.67934×10^{-2}
0.9	0.01	11.66714401	7.06524×10^{-4}
0.9	0.001	11.66784353	7.00713×10^{-6}
0.9	0.0001	11.66785047	7.00248×10^{-8}

Example 3

Consider the fractional differential equation:

$y^{(0.5)} + y = t^2 + 2\,t^{\frac{3}{2}}$, $y(0.1) = 0.01$, $t \in [0.1,1]$.

According to (13), table (3) and table(4) show the approximation of $y(t)$ at $t_i = 0.1 + h * i$ for $i = 1,2,\dots,10$.

Table 3
Approximation of $y(t)$ of example 3 according to (13) where h=0.1.

x_i	Approximation	Absolute Error
0.1	0.01	0
0.2	0.03	1×10^{-2}
0.3	0.0722361	1.77639×10^{-2}
0.4	0.135479	2.45207×10^{-2}
0.5	0.219356	3.06436×10^{-2}
0.6	0.32369	3.631×10^{-2}
0.7	0.448378	4.16224×10^{-2}
0.8	0.593352	4.66475×10^{-2}
0.9	0.758568	5.14322×10^{-2}
1	0.943989	5.60108×10^{-2}

Table 4
Approximation of $y(t)$ of example 3 according to (13) where h=0.01.

x_i	Approximation	Absolute Error
0.1	0.01	0
0.2	0.0391048	8.95158×10^{-4}
0.3	0.0883543	1.64567×10^{-3}
0.4	0.1576861	2.31387×10^{-3}
0.5	0.2470745	2.92541×10^{-3}
0.6	0.3565058	3.49418×10^{-3}
0.7	0.4859710	4.02891×10^{-3}
0.8	0.6354644	4.53553×10^{-3}
0.9	0.8049816	5.01833×10^{-3}
1	0.9945194	5.48056×10^{-3}

REFERENCES

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