DESIGN OF ROBUST CONTROLLERS FOR UNCERTAIN PIECEWISE LINEAR SYSTEMS

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ABSTRACT

A systematic method to design robust digital controllers for uncertain piecewise linear systems with structured parametric uncertainties is presented in this paper. This method consists of decomposing the uncertainties of the original system such that it can be rewritten as a set of local linear models with additional disturbed inputs. For this representation of the uncertain system, a continuous-time robust controller is designed as the solution of a linear matrix inequality problem with pole placement restrictions. The robust digital controller is obtained as the digital redesign of the continuous-time robust controller in the state-matching sense. Numerical solution of the tracking problem for the uncertain chaotic Chua's circuit is used to illustrate the effectiveness of the proposed method.

1. INTRODUCTION

In general, a real world system can be suitably represented as a continuous-time parametric uncertain nonlinear model with bounded disturbances and noise inputs [11]. However, it is common practice to describe a real system using linear models; one motivation to use this type of models is the possibility of using well-developed linear techniques for analysis and controller design [13, 14]. Unfortunately this approach is only valid for the limited range of operation where the original dynamics do not deviate too far from the linear model. Therefore, when the system experiences uncertainties, modeling errors and other disturbances, the linear approach usually fails to produce satisfactory results.

The main objective of robust control is to design feedback controllers that achieve the control objectives even in the presence of these types of perturbations [5]. The design of robust controllers for continuous-time uncertain plants has been an active research topic for many years; see for example [4, 5]. However, most of the research efforts have been concentrated on the design of continuous-time controllers for continuous-time plants or discrete-time controllers for discrete-time systems [1, 2, 10]. In this paper the hybrid case is considered; our objective is to design a discrete-time robust controller, which can be implemented on digital devices, for a continuous-time uncertain piecewise-linear plant with structured uncertainties.

In particular, the case of parametric uncertainties is considered, this type of uncertainty is structured [11], with this in mind, it can be decomposed and considered as an extra disturbance input, for this alternative representation, a continuous-time robust controller can be designed as the solution of a linear matrix inequality problem [4, 12].

To obtain a robust controller suitable for implementation on digital devices the state-matching digital redesign approach is considered [6, 7, 8, 9]. With this methodology a continuous-time controller previously designed to satisfy a set of control objectives, like robustness in this case, is used to determine a discrete-time controller such that when applied to the continuous-time plant the states of the hybrid controlled system, match at least on every sampling instant the states of the continuous-time controlled system originally designed.

During the last decade Shieh et al. had extensively study this methodology and have proposed different digital redesign techniques [13, 14, 15], which had been applied to different types of systems, including uncertain linear systems [16, 17, 18, 19, 20]. However, the application of digital redesign techniques to uncertain nonlinear systems has not been reported.

The rest of the paper is organized as follows: In Section 2, a method is presented to decompose the structured uncertainties of the original system such that it can be represented as a set of nominal linear systems with an extra disturbance input. Then, for this alternative representation, a continuous-time robust controller is designed using the linear matrix inequality approach. In Section 3, the Chebyshev quadrature method of state matching digital redesign is presented as a way to obtain a robust discrete-time controller for the uncertain piecewise linear system. To illustrate the effectiveness of the proposed method, in Section 4, numerical simulations of a discrete-time controller that solves the tracking problem for an uncertain chaotic Chua's circuit is presented. In Section 5, some comments and conclusions are stated.

2. CONTINUOUS-TIME ROBUST CONTROLLER DESIGN

Consider the uncertain piecewise linear system:

$$\dot{x}_c(t) = \tilde{A}_j(\rho)x_c(t) + \tilde{B}_j(\rho)u_c(t) \qquad (1)$$

for $j = 1,2,...,q$, where $x_c(t) \in \Re^n$ represents the states of the system, $u_c(t) \in \Re^m$ is the control input and $\rho \in \Re^p$ represents the system parameters, while are unknown but bounded to a given interval of possible values, such that the uncertain matrices of (1) can be written as:

$$\tilde{A}_j(\rho) = A_{o,j} \pm \Delta A_j \quad (2a)$$

$$\tilde{B}_j(\rho) = B_{o,j} \pm \Delta B_j \quad (2b)$$

where $A_{o,j} \in \Re^{n\times n}$ and $B_{o,j} \in \Re^{n\times m}$ are the nominal system and nominal input matrices respectably, while $\Delta A_j \in \Re^{n\times n}$ and $\Delta B_j \in \Re^{n\times m}$ are unknown but bounded structured uncertainty matrices corresponding to the effects of the parameter uncertainties.

The uncertainty matrices can be rewritten in terms of the uncertain elements $(\Delta_{a_l}, \Delta_{b_l})$ and the constant matrices $A_{j,l} \in \Re^{n\times n}$ and $B_{j,l} \in \Re^{n\times m}$ as:

$$\Delta A_j = \sum_{l=1}^{k_a} \Delta_{a_l} A_{j,l} = M_{j,ac} \Delta_{j,a} N_{j,ar} \quad (3a)$$

$$\Delta B_j = \sum_{i=1}^{k_b} \Delta_{b_l} B_{j,l} = M_{j,bc} \Delta_{j,b} N_{j,br} \quad (3b)$$

From this representation by letting $q_l \overset{\Delta}{=} rank(A_{j,l})$ and $p_l \overset{\Delta}{=} rank(B_{j,l})$, , the constant matrices $M_{j,ac}$, $M_{j,bc}$, $N_{j,ar}$ and $N_{j,br}$ are given by the equations [20]

$$M_{j,ac} = [M_{ac,1},\ M_{ac,2},\ \dots, M_{ac,k_a}], \quad (4a)$$

$$M_{j,bc} = [M_{bc,1},\ M_{bc,2},\ \dots, M_{bc,k_b}], \quad (4b)$$

$$N_{j,ar} = \begin{bmatrix} N_{ar,1}^T & N_{ar,2}^T & \dots & N_{ar,k_a}^T \end{bmatrix}^T \quad (4c)$$

$$N_{j,br} = \begin{bmatrix} N_{br,1}^T & N_{br,2}^T & \dots & N_{br,k_b}^T \end{bmatrix}^T \quad (4d)$$

where $M_{ac,l} \in \Re^{n\times q_l}$ are the q_l nonzero column vectors of $A_{j,l}$, $M_{bc,l} \in \Re^{n\times p_l}$ are the p_l nonzero column vectors of $B_{j,l}$, while

$$N_{ar,l} = (M_{ac,l}^T M_{ac,l})^{-1} M_{ac,l}^T A_{j,l} \in \Re^{q_l \times n} \quad (5a)$$

$$N_{br,l} = (M_{bc,l}^T M_{bc,l})^{-1} M_{bc,l}^T B_{j,l} \in \Re^{p_l \times m} \quad (5b)$$

with

$$\Delta_{j,a} = block\,diag[\Delta_{a_1} I_{q_1},\ \Delta_{a_2} I_{q_2},\ \dots, \Delta_{a_{k_b}} I_{q_{k_b}}]$$

$$\Delta_{j,b} = block\,diag[\Delta_{b_1} I_{p_1},\ \Delta_{b_2} I_{p_2},\ \dots, \Delta_{b_{k_b}} I_{p_{k_b}}].$$

Here I_{q_l} and I_{p_l}, represents the $q_l \times q_l$ and $p_l \times p_l$ identity matrices respectively.

Without loss of generality, one can assume that $\left|\Delta_{a_i}\right| \le 1$ and $\left|\Delta_{b_j}\right| \le 1$ for $i = 1, \dots, k_a$; $j = 1, \dots, k_b$. So that the uncertain piecewise linear system (1) can be rewritten as a set of nominal linear system plus one fictitious disturbed input, in the form:

$$\begin{pmatrix} \dot{x}_c(t) \\ z_c(t) \\ y_c(t) \end{pmatrix} = \begin{pmatrix} A_{o,j} & B_{I,j} & B_{o,j} \\ C_{I,j} & 0 & D_{I,J} \\ C_{o,j} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_c(t) \\ \bar{w}_c(t) \\ u_c(t) \end{pmatrix} \quad (6)$$

for $j = 1,2,...,q$, where $B_{I,j} = \begin{bmatrix} M_{j,ac}, M_{j,bc} \end{bmatrix}$, $C_{I,j} = \begin{bmatrix} N_{j,ar}^T & 0 \end{bmatrix}^T$, $D_{I,j} = \begin{bmatrix} 0 & N_{j,br}^T \end{bmatrix}^T$ and $C_{o,j} = I_n$.

With the disturbed input given by

$$\bar{w}_c(t) = diag\ \begin{bmatrix} \Delta_{j,a}, & \Delta_{j,b} \end{bmatrix} z_c(t).$$

The objective is to construct a static feedback controller for the disturbed system (6) in the form

$$u_c(t) = K\, y_c(t) \quad (7)$$

such that in closed-loop the system (6)-(7) be internally stable and the effects of the disturbed input $\bar{w}_c(t)$ on the desired output $z_c(t)$ measured as the infinity norm of the transfer function $\hat{T}_{z_c \bar{w}_c}(s)$, be less than a given bound $\gamma > 0$.

In closed-loop the system becomes:

$$\begin{pmatrix} \dot{x}_c(t) \\ z_c(t) \end{pmatrix} = \begin{pmatrix} A_{Cl} & B_{Cl} \\ C_{Cl} & D_{Cl} \end{pmatrix} \begin{pmatrix} x_c(t) \\ \bar{w}_c(t) \end{pmatrix} \quad (8)$$

where $A_{Cl} = A_{o,j} + B_{o,j} K$, $B_{Cl} = B_{I,j}$, $C_{Cl} = C_{I,j} + D_{I,J} D_{\chi,j}$. $D_{Cl} = 0$,

For a perturbed system, the property of stability is called internal stability and is defined as follows.

Definition 1: Consider the general feedback system shown in Figure 1.

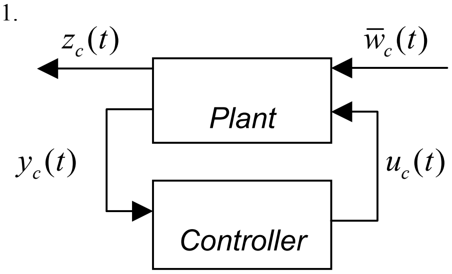

Figure 1 *General feedback system*

This system is internally stable if for every initial condition $x_C(0)$ of the plant G, and $x_K(0)$ of controller K, the limits:

$$\lim_{t\to\infty} x(t) \to 0,$$

$$\lim_{t\to\infty} x_K(t) \to 0,$$

hold when $w(t)=0$.

A necessary and sufficient condition for the existence of an internally stabilizing controller K for the system in (8) is that the plant $(A_{o,j}, B_{o,j}, C_{o,j})$ is both stabilizable and detectable. The simplest stabilizing controller that can be designed is a static state feedback, which reduces the stabilizing controller K to $D_K = F$, where the matrix gain F must be designed such that $A_{o,j} + B_{o,j}F$ is Hurwitz. However, other control objectives might need to be considered in addition to the internal stability such as minimizing a performance criterion expressed in terms of the H_2 or H_∞ norms of the transfer function between an unknown disturbance input and the controlled output.

These norms of a transfer function are defined as follows:

$$\|\hat{T}(s)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr[(\hat{T}(j\omega))^* \hat{T}(j\omega)]d\omega \tag{9}$$

$$\|\hat{T}(s)\|_\infty = \sup_\omega \sigma_{\max}[\hat{T}(j\omega)] \tag{10}$$

where $tr[\cdot]$ and $\sup[\cdot]$ indicate the trace and supremum operations, $(\cdot)^*$ represents the Hermitian matrix and $\sigma_{\max}$ indicates the maximum singular value of the matrix function.

H_2 or H_∞ norms can also be calculated directly from the following Lemma:

Lemma 1: [5].

Consider a transfer function $\hat{T}(s)$ with the state space realization

$$\hat{T}(s) = \left[\begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & 0 \end{array}\right], \tag{11}$$

with T_{11} a Hurwitz matrix. Then

$$\|\hat{T}(s)\|_2^2 = tr\left[T_{21}\ \Psi_c\ (T_{21})^*\right] = tr\left[(T_{12})^*\ \Psi_o\ T_{12}\right] \tag{12}$$

where

$$T_{11}\Psi_c + \Psi_c(T_{11})^* + T_{12}\ (T_{12})^* = 0,$$

$$(T_{11})^*\Psi_o + \Psi_o\ T_{11} + (T_{21})^* T_{21} = 0.$$

The matrices Ψ_c and Ψ_o are the controllability and observability Gramian, respectively.

The norm $\|\hat{T}(s)\|_\infty$ is given in relation to a bound $\gamma > 0$ obtained in reference to the Hamiltonian matrix define as

$$\hat{H} := \begin{bmatrix} T_{11} & \frac{1}{\gamma^2}\ T_{12}\ (T_{12})^* \\ -(T_{21})^*\ T_{21} & -(T_{11})^* \end{bmatrix} \tag{13}$$

Then, the following conditions are equivalent:

a) $\|\hat{T}(s)\|_\infty < \gamma$,

b) $\hat{H}$ has no eigenvalues on the imaginary axis,

c) $\hat{H} \in dom(Ric)$,

d) $\hat{H} \in dom(Ric)$ and $Ric(\hat{H}) \geq 0$ ($Ric(\hat{H}) > 0$ if (T_{11}, T_{21}) is observable),

where $X = Ric(\hat{H})$ is called the Riccati operator which represents the solution to the Riccati equation

$$T_{11}^T X + XT_{11} + X\frac{1}{\gamma^2}\ T_{12}\ (T_{12})^* X + (T_{21})^* = 0,$$

associated with the Hamiltonian matrix (13) such that $T_{11} + \frac{1}{\gamma^2}\ T_{12}\ (T_{12})^* X$ is stable. The matrices $\hat{H}$ for which $X = Ric(\hat{H})$ exists is the domain of the Riccati operator and is denoted by $dom(Ric)$.

The problem of designing a robust controller with H_∞ performance criterion can be described as follows:

Definition 2: The H_∞ robust control problem is to find a feedback controller K of the form (8) for a given nominal state space plant G (6), such that the closed-loop system becomes internally stable and the transfer function $\hat{T}_{z_C\bar{w}_C}(s)$, from the disturbance input $\bar{w}_C(t)$ to the desired output $z_c(t)$ satisfies the performance index $\|\hat{T}_{z_c\bar{w}_c}(s)\|_\infty < \gamma$ with same pre-assigned $\gamma > 0$.

The solution of the above problem can be derived using linear matrix inequality (LMI) techniques as described next.

Assuming that system (8) is both controllable and observable, the following well-known result of robust control theory can be used to determine the controller.

Lemma 2: [5] Kalman-Yacubovich-Popov Lemma (KYP).

Suppose the transfer function is $\hat{M}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Then, the following are equivalent conditions:

a) The matrix A is Hurwitz and $\|\hat{M}(s)\|_\infty < 1$;

b) There exist a matrix $X > 0$ such that

$$\begin{bmatrix} C^* \\ D^* \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^*X + XA & XB \\ B^*X & -I \end{bmatrix} < 0.$$

Following the KYP lemma, an H_∞ controller that satisfies the requirements stated in Definition 2 can be found if there exists a solution $\hat{\mathrm{X}} = \hat{\mathrm{X}}^T$ to the following matrix inequality:

$$\begin{pmatrix} A_{Cl}^T\hat{X}+\hat{X}A_{Cl} & \hat{X}B_{Cl} \\ B_{Cl}^T\hat{X} & 0 \end{pmatrix}+ \quad , \tag{14}$$
$$\begin{pmatrix} 0 & I \\ C_{Cl} & D_{Cl} \end{pmatrix}^T \begin{pmatrix} -\gamma I & 0 \\ 0 & (\gamma I)^{-1} \end{pmatrix} \begin{pmatrix} 0 & I \\ C_{Cl} & D_{Cl} \end{pmatrix} < 0$$

where the matrices are defined as in (8).

Obviously, the controller cannot be obtained directly from (14) since the inequality depends nonlinearly on the variables $\hat{X}$ and $(A_K,\ B_K,\ C_K,\ D_K)$. However, one may use a nonlinear transformation of the form

$$\left(\hat{X}, \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}\right) \rightarrow \upsilon = \left(X, Y, \begin{pmatrix} K & L \\ M & N \end{pmatrix}\right), \tag{15}$$

to transform the nonlinear matrix inequality (15) to a matrix inequality with an affine structure on the variable υ, such that $X(\upsilon) > 0$,

$$\begin{pmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} A(\upsilon) & B(\upsilon) \\ C(\upsilon) & D(\upsilon) \end{bmatrix} \end{pmatrix}^T \quad , \tag{16}$$
$$\begin{pmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -\gamma I \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & (\gamma I)^{-1} \end{bmatrix} \end{pmatrix} \begin{pmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} A(\upsilon) & B(\upsilon) \\ C(\upsilon) & D(\upsilon) \end{bmatrix} \end{pmatrix} < 0$$

where $X(\upsilon) = \begin{bmatrix} Y & I \\ I & X \end{bmatrix}$ and

$$\begin{pmatrix} A(\upsilon) & B(\upsilon) \\ C(\upsilon) & D(\upsilon) \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} A_{o,j}Y + B_{o,j}M & A_{o,j} + B_{o,j}NC_{o,j} \\ K & A_{o,j}X + LC_{o,j} \end{bmatrix} & \begin{bmatrix} B_{I,j} \\ XB_{I,j} \end{bmatrix} \\ \begin{bmatrix} C_{I,j}Y + D_{I,j}M & C_{I,j} + D_{I,j}NC_{o,j} \end{bmatrix} & 0 \end{pmatrix} \tag{17}$$

The matrix inequality in (16) is still not linear, so it is necessary to restructure the matrix inequality using the Schur complement argument [5], which yields the following linear matrix inequalities:

$$\begin{pmatrix} A(\upsilon)^T + A(\upsilon) & B(\upsilon) & C(\upsilon)^T \\ B(\upsilon)^T & -\gamma I & D(\upsilon)^T \\ C(\upsilon) & D(\upsilon) & -\gamma I \end{pmatrix} < 0,\ Y > 0 \tag{18}$$

where $A(\upsilon) = A_{o,j}Y + B_{o,j}N$, $B(\upsilon) = B_{I,j}$, $C(\upsilon) = C_{I,j}Y + D_{I,j}N$, $D(\upsilon) = D_{Cl} = 0$ and $Y = \hat{X}^{-1}$.

Which can be solved for the matrix variables Y and N.

A further requirement that can be imposed on the robust controller is to have a fast and well-damped time response. This is can be accomplish solving a regional pole placement problem as follows:

To assign all the closed-loop poles of the system to the LMI region $\mathbb{D}$, (Chalali et. al., 1999):

$$\mathbb{D} = \left\{ z \in \mathbb{C} \middle| L + zM + \bar{z}M^T < 0 \right\} \tag{19}$$

where $L = L^T$ and M are real matrices. For a disk centered at $(-q, 0)$ with radius r the LMI region is characterized by

$$f_D(z) = \begin{bmatrix} -r & q \\ q & -r \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \bar{z}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T \tag{20}$$

The real matrix $A(v)$ has all its eigenvalues in the LMI region $\mathbb{D}$, if and only if a symmetric matrix $Y > 0$ exists such that

$$\begin{gathered} L \otimes Y + M \otimes \left(YA(v)\right) + M^T \otimes \left(A(v)^T Y\right) < 0 \\ \left[l_{ij}Y + m_{ij}\left(YA(v)\right) + m_{ji}\left(A(v)^T Y\right) \right]_{1 \le i,j \le 0} < 0 \end{gathered} \tag{21}$$

where $\otimes$ is the Kronecker product, l_{ij} and m_{ij} are the i, j th entries of the matrices L and M, respectably.

In particular, if the control objective is for system (1) to track the reference orbit, $x_r(t)$, and locating the closed-loop poles on a disk on the left half complex plane. Then the controller can be found solving the linear matrix inequalities (11) and (14) in terms of the tracking error variable

$$x_e(t) = x_c(t) - x_r(t) \tag{22}$$

Then, the robust static state feedback tracking controller that makes the closed-loop system internally stable with closed-loop poles on the disk $(center = -q, radius = r)$, and satisfies the performance index $\left\| \hat{T}_{z_c \bar{w}_c}(s) \right\|_\infty < \gamma$ is given by

$$u_c(t) = -K_{c,j}x_e(t) \tag{23}$$

where the feedback gains are $K_{c,j} = -N_j Y^{-1}$ for each j subsystem.

3. DIGITAL REDESIGN VIA CHEBYSHEV FORMULA

Digital redesign can be defined as the process of converting a previously well-designed continuous-time controller to a discrete-time controller suitable for digital implementation, such that the states of the continuous-time and sampled-data closed-loop systems match at least at every sampling instant for the entire process.

Consider a controllable and observable continuous-time linear system

$$\dot{x}_c(t) = A_{o,j}x_c(t) + B_{o,j}u_c(t) \tag{24}$$

where $x_c(t) \in \Re^n$, $u_c(t) \in \Re^m$, $(A_{o,j}, B_{o,j})$ are constant matrices of appropriate dimensions with (12) the continuous-time control law.

In closed loop the continuous-time system becomes

$$\dot{x}_c(t) = \left(A_{o,j} - B_{o,j}K_{c,j}\right)x_c(t) \tag{25}$$

a discrete-time representation of this system can be found from the solution to the system at each sampling instant as

$$x_c(kT+T) = G_{C,j}\, x_c(kT) \tag{26}$$

where $G_{C,j} = e^{A_{C,j}T}$ and $A_{C,j} = A_{o,j} - B_{o,j}\, K_{C,j}$.

The objective of digital redesign is to find a discrete-time controller in the form

$$u_d(kT) = -K_{d,j} x_d(kT) \tag{27}$$

where $T > 0$ is the sampled-hold period and $K_{d,j}$ is the digital feedback gain, such that the states, $x_d(t)$, of the closed-loop hybrid controlled system

$$\dot{x}_d(t) = A_{o,j} x_d(t) + B_{o,j}\left(-K_{d,j}\, x_d(kT)\right) \tag{28}$$

match the states, $x_c(t)$ of the continuous-time control system.

The assumption that discrete-time controller (27) is applied to the continuous-time system (24) in place of (22) using a zero order hold device, in this way the control law applied is piecewise constant, such that

$$u_d(t) = u_d(kT) \tag{29}$$

for $kT < t < kT + T$.

With controller (29) a discrete-time model of the system (28) can be obtain as

$$x_d(kT+T) = G_j\, x_d(kT) - H_j\, u_d(kT) \tag{30}$$

with $G_j = e^{A_{o,j}T}$ and $H_j = (G_j - I_n) A_{o,j}^{-1} B_{o,j}$.

The gains $K_{d,j}$ that constitute the discrete-time control law can be obtained from the values of the continuous-time gains $K_{c,j}$ using the general Chebyshev quadrature approximation formula:

$$\int_a^b w(\lambda) f(\lambda) d\lambda \cong W \sum_{i=0}^{N} f(\lambda_i) \tag{31}$$

where $w(\lambda)$ is a constant sign weighting function in $[a,b]$, W is the weighting factor determined by $W = \frac{1}{(N+1)} \int_a^b w(\lambda) d\lambda$, and $f(\lambda_i)$ are the values of the function $f(\lambda)$ evaluated at $\lambda = \frac{a + i\ (b-a)}{N}$, for $i = 0,1,2,...,N$.

The solution of the continuous-time closed-loop system at the sampling instant $kT + T$ can be written as

$$x_c(kT+T) = G_j\, x_c(k) - \int_{kT}^{kT+T} e^{A_{o,j}(kT+T-\tau)} B_{o,j} K_{c,j} x(\tau) d\tau \tag{32}$$

The resulting matrix convolution integral can be solved by approximations using the Chebyshev quadrature formula:

$$\int_{kT}^{kT+T} e^{A_{o,j}(kT+T-\tau)} B_{o,j} K_{c,j} x(\tau) d\tau = \frac{1}{N+1} H_j K_{c,j} \sum_{i=0}^{N} x_c(k + iT_N) \tag{33}$$

where $T_N \equiv \frac{T}{N}$ and $x_c(k + iT_N)$ represents the intersampled states of the continuous-time system, which are given by

$$x_c(kT + iT_N) = G_{c,j}^{(i)}\, x_c(kT) \tag{34}$$

where $G_{C,j}^{(i)} = \left(G_j\right)^i = e^{A_{C,j} T_N i}$.

Using these results the discrete-time model of the continuous-time closed-loop system becomes

$$x_c(kT+T) = G_j\, x_c(kT) - H_j\left[\frac{1}{N+1} K_{c,j} \sum_{i=0}^{N} G_{c,j}^{(i)}\right] x_c(kT) \tag{35}$$

To satisfy the matching requirement from (35) and (30), taking into account the intersampled states $x_C(k + iT_N)$ for $i = 0,1,2,...,N,$ one must have

$$G_j - H_j K_{d,j} = G_j - H_j\left[\frac{1}{N+1} K_{c,j} \sum_{i=0}^{N} G_{c,j}^{(i)}\right] \tag{36}$$

Solving (27) for the discrete-time gains $K_{d,j}$ when $N \to \infty$, one obtains

$$K_{d,j} = K_{c,j}\left(A_{c,j} T\right)^{-1}\left(G_{c,j} - I_n\right) \tag{37}$$

4. NUMERICAL RESULTS

On its dimensionless form Chua's circuit is given by the equations

$$\begin{aligned} \dot{x}_1(t) &= \alpha\left[x_2(t) - x_1(t) - g_{NL}(x_1(t))\right] \\ \dot{x}_2(t) &= x_1(t) - x_2(t) + x_3(t) \\ \dot{x}_3(t) &= -\beta x_2(t) \end{aligned} \tag{38}$$

where $g_{NL}\left(x_1(t)\right) = g_b x_1(t) + \frac{1}{2}\left(g_a - g_b\right) v(t)$, with $v(t) = \left|x_1(t)+1\right| - \left|x_1(t)-1\right|$.

This circuit can be rewritten as a piecewise linear system in the following form (Chen and Dong, 1998):

$$\dot{x}_c(t) = \tilde{A}_j\, x_c(t) + \tilde{B}_j\, u_{c,j}(t)\ ,\ j = 1,2,3 \tag{39}$$

where

$$\tilde{A}_1 = \tilde{A}_3 = \begin{bmatrix} -\tilde{\alpha}\left(1+g_a\right) & \tilde{\alpha} & 0 \\ 1 & -1 & 1 \\ 0 & -\tilde{\beta} & 0 \end{bmatrix},$$

$$\tilde{A}_2 = \begin{bmatrix} -\tilde{\alpha}\left(1+g_b\right) & \tilde{\alpha} & 0 \\ 1 & -1 & 1 \\ 0 & -\tilde{\beta} & 0 \end{bmatrix},$$

$$\tilde{B}_1 = \begin{bmatrix} -\tilde{\alpha}\left(g_b-g_a\right) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; \tilde{B}_2 = \tilde{B} I_{3\times 3} \text{ and}$$

$$\tilde{B}_3 = \begin{bmatrix} \tilde{\alpha}\left(g_b-g_a\right) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $u_{c,1} = u_{c,3} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $u_{c,2}(t) = 0_{3\times 3}$.

Assume that the parameters of (39) are uncertain but bound to a given interval, such that

$\tilde{\alpha} = \alpha_0 + \Delta\alpha_I$, $\tilde{\beta} = \beta_0 + \Delta\beta_I$ $\tilde{B} = B_0 + \Delta B_I$

The control objective is to track the reference orbit given by (Chen & Dong,1993):

$$\begin{aligned} x_{r1}(t) &= a\ \cos(\xi)\cos(wt) - b\ \sin(\xi)\sin(wt) + \\ &\quad c\ \cos(\xi)\cos(2wt) - d\ \sin(\xi)\sin(2wt) \\ x_{r2}(t) &= e\left[a\ \sin(\xi)\cos(wt) - b\ \cos(\xi)\sin(wt)\right] + \\ &\quad f\ \left[c\ \sin(\xi)\cos(2wt) - d\ \cos(\xi)\sin(2wt)\right] \\ \dot{x}_{r3}(t) &= -\beta\ \ x_{r2}(t) \end{aligned} \tag{40}$$

Then, a robust tracking controller can be constructed in terms of the tracking error $x_{e,j}(t) = x_{c,j}(t) - x_{r,j}(t)$ decomposing the uncertainties of (39) into the matrices

$$M^*_{ac,1} = M^*_{ac,3} = \begin{bmatrix} -\alpha_I\left(1+g_a\right) & \alpha_I \\ 0 & 0 \\ 0 & -\beta_I \end{bmatrix},$$

$$M^*_{ac,2} = \begin{bmatrix} -\alpha_I\left(1+g_b\right) & \alpha_I \\ 0 & 0 \\ 0 & -\beta_I \end{bmatrix},$$

$$N^*_{ar,1} = N^*_{ar,2} = N^*_{ar,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$M^*_{bc,1} = \begin{bmatrix} -\alpha_I\left(g_b-g_a\right) \\ 0 \\ 0 \end{bmatrix}, \; M^*_{bc,2} = \begin{bmatrix} B_I \\ B_I \\ B_I \end{bmatrix},$$

$$M^*_{bc,3} = \begin{bmatrix} \alpha_I\left(g_b-g_a\right) \\ 0 \\ 0 \end{bmatrix} \text{ and}$$

$N^*_{br,1} = N^*_{br,2} = N^*_{br,3} = 1$.

The numerical simulations were carried out on SimuLink of MatLab using a fifth order Dormand-Prince algorithm with a fixed integration step $\tau = 0.001$ for the parameter set $\alpha_0 = 9$, $\beta_0 = 100/7$, $\alpha_I = 45/100$, $\beta_I = 5/7$, $g_a = -5/7$ and $g_b = -8/7$.

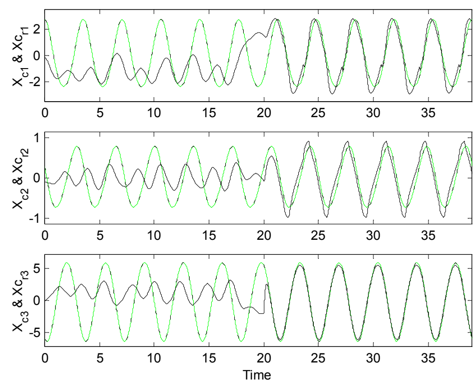

Figure 2. *Performance of the digitally redesigned robust tracker for Chua's circuit with* $\gamma = 0.1$ *and* $T = 0.1$ *(*x_c *solid,* x_r *dotted).*

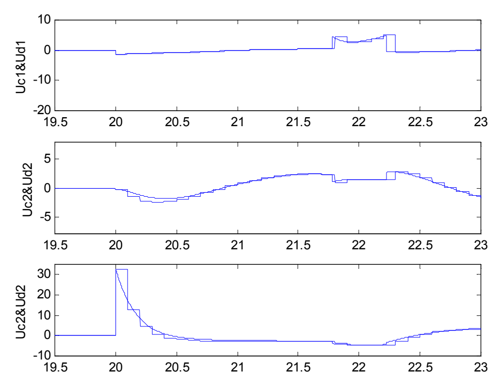

Figure 3. *Comparison between the continuous-time and the digitally redesigned control law for tracking on Chua's circuit* $\gamma = 0.1$ *and* $T = 0.1$ *(*U_c *smooth,* U_d *stair like).*

The results of applying the digitally redesigned robust controller for a performance bound of $\gamma = 0.1$ with pole location on the disk $\left(q = -9, r = 5\right)$ and a sampled-hold period of $T = 0.1$, after twenty seconds of chaotic behavior of the uncertain Chua's circuit are presented on Figure 2. On Figure 3, the continuous-time and digitally redesign control laws are presented for comparison purposes

5. CONCLUSIONS

A systematic method to design robust controller suitable for digital implementation that can solve the tracking problem for an uncertain piecewise linear systems has been presented. The new method consists of three steps: Decomposition of the structured uncertainty, continuous-time robust design and digital redesign. One of the advantages of this approach is that the discrete-time implementation maintains comparable robust performance with respect to the original continuous-time design while allowing for significantly larger sampled-hold periods comparing it to a direct implementation of the continuous-time controller.

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