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THE ROBUST LINKAGE SYNTHESIS FOR PLANAR RIGID-BODY GUIDANCE

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ABSTRACT

The kinematic synthesis of planar motion generators in the presence of an incomplete set of finitely separated poses is the subject of this paper. Given that the planar rigid-body guidance problem in the realm of four-bar linkage synthesis can be solved exactly for up to five prescribed poses of the coupler link, any number of poses smaller than five is considered incomplete in this paper. The poses completing the set are determined so as to produce a robust linkage against variations in the unspecified poses. To this end, a theoretical framework for model-based robust design is invoked and a general methodology for robust kinematic synthesis is laid down. Robustness is needed in this context to overcome the presence of uncertainty due to the selection of the unprescribed poses, which many a time are left up to the mechanism designer's judgment. To validate the concepts and illustrate the application of the methodology proposed here, an example is included.

1 INTRODUCTION

The kinematic synthesis of linkages is at the core of mechanical design. In many cases, the mechanical designer encounters situations in which it is necessary to guide a rigid body through a discrete set of finitely separated poses—positions and orientations. We recall that the motion-generation problem of planar four-bar linkages can be solved *exactly* for up to five prescribed poses [1]. When the number of prescribed poses is five, we consider the set of poses *complete*. However, solutions for less than the said pose number lead to underdetermined problems associated with incomplete pose sets, which are rendered determinate by, usually, specifying *arbitrarily* the intermediate poses. Granted, simple techniques are available to synthesize the planar linkages for two and three poses [2], but these methods are rather limiting in that they rely on the arbitrary location of some joint centers. This decision is left up to the designer, who bases her or his decision on both experience and space limitations. The latter invariably involves a bounded region, which offers a non-denumerable set of possibilities.

In fact, the problem of interest can be solved numerically within the formalism of the classical Burmester problem [1], whose solution relies on a well-established methodology, developed for the most challenging cases of four and five poses. Nevertheless, this formalism requires additional intermediate poses, which are usually left up to the discretion of the linkage designer. Typically in a design task, the unspecified poses would allow the designer to prescribe some coordinates of the center and circle-points, and would do so by gut feeling. Hence, a rigid-body guidance problem assigned to N different expert designers will most likely lead to N different sets of linkages that, most likely, will be quite disparate. Given the arbitrariness of the selection of the unspecified intermediate poses to define an exact linkage, the designer of the system at hand faces a problem of uncertainty. To cope with this uncertainty, we resort to the philosophy of robust design, as proposed by Taguchi and his school [3, 4]. The choice of those poses so as to produce a *robust linkage* is discussed here.

We start by recalling the theoretical foundations, and then propose a methodology applicable to the robust selection of the unspecified poses, besides the dimensions of the planar four-bar linkage used to execute the task at hand.

Generally speaking, robust design is based on a philosophy that enables the design of products whose performance is minimally sensitive to unavoidable variations in environment conditions [3]. Therefore, products robustly designed can perform well over a broad range of variations of the environment in which the product will operate [5]. In fact, the need to design products which are as insensitive as possible to uncontrollable changes is of growing interest and recognized importance. As a consequence, over the past decades, robust design has attracted the attention of practitioners and researchers in many engineering fields. In connection with mechanism design, some researchers have reported on the use of orthogonal arrays for synthesizing linkages [6]. Other researchers, in turn, have engaged in extending Taguchi's method to allow for constrained mechanism dimensional synthesis [7]. On the other hand, the validity of Taguchi's method for robust design has been critically scrutinized. With the aid of nonlinear programming, the problem of planar-linkage robust design has been formulated as the minimization of the weighted root mean square (rms) of the errors [8].

In fact, the exact synthesis of four-bar linkages intended for rigid-body guidance has been studied both from the theoretical and from the computational viewpoints [1,9–11]. Work has been reported on the production of graphics software packages for mechanism design [12,13].

Actually, the exact synthesis problem of planar four-bar linkages for a complete number of five poses can be reduced to solving a quartic univariate polynomial [1,11]. This polynomial admits four roots, of which zero, two, or four will be real. Thus, there can be as many as four RR dyads that reach five poses exactly. Now, the number of combinations of four objects taking two at time is six, which provides an upper bound on the number of linkages capable of visiting exactly five given poses. That is, up to six different linkages can guide a rigid body through five poses. Robustness was proposed in [14] as a criterion to choose one out of those six linkages, which gives a rather limited set of possible choices. We propose here an alternative approach that searches for the robust linkage within a non-denumerable set of possible candidates. This paper is an extension and enhancement of the work reported in [15] on the robust synthesis of planar four-bar linkages.

It is worth mentioning that even though this paper targets the synthesis problem for one class of tasks, namely, rigid-body guidance, the formulation included here is general and can be extended to other tasks.

2 A ROBUST DESIGN FRAMEWORK

The methodology we will adopt here for guaranteeing robustness is based on a modern approach for model-based robust engineering design, as introduced in [16]. For completeness, we recall below the main concepts and the fundamental results reported therein.

2.1 Performance Functions and their Arguments

The various quantities involved in a design task can be classified into three sets: (*i*) *design variables* (DV), which are to be assigned values by the designer so as to meet performance specifications under given conditions, (*ii*) *design-environment parameters* (DEP), over which the designer has no control, and that define the conditions of the environment under which the designed object will operate; and (*iii*) *performance functions* (PF), representing the performance of the design in terms of design variables and design-environment parameters. Henceforth, we shall denote by $\mathbf{x}$ the *n*-dimensional vector of DV. Likewise, we shall denote by $\mathbf{p}$ the ν-dimensional vector of DEP, while the *m-performance functions*, $f_i = f_i(\mathbf{x};\mathbf{p})$, for $i = 1,2,\ldots,m$, are grouped in the *performance function vector* (PFV) $\mathbf{f}$. We thus have

$$\mathbf{x} \equiv \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{p} \equiv \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_\nu \end{bmatrix}, \quad \mathbf{f} \equiv \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}, \quad \mathbf{f} = \mathbf{f}(\mathbf{x};\mathbf{p}) \tag{1}$$

2.2 Robustness Index

Robust design aims at rendering the performance vector $\mathbf{f}$ of a design as insensitive to the variations $\Delta\mathbf{p}$ as possible. In this vein, we assume that the functional relation $\mathbf{f}(\mathbf{x};\mathbf{p})$ of Eqn. (1) is differentiable with respect to the DEP, and hence, we have, in light of Eqn. (1), and upon expansion around the *nominal point* $(\mathbf{x},\mathbf{p}_0)$, for a fixed $\mathbf{x}$, while considering only first-order terms,

$$\Delta\mathbf{f} = \mathbf{F}\Delta\mathbf{p}, \quad \mathbf{F}(\mathbf{x};\mathbf{p}_0) \equiv \left.\frac{\partial \mathbf{f}}{\partial \mathbf{p}}\right|_{\mathbf{p}=\mathbf{p}_0} \in \mathbb{R}^{m\times\nu} \tag{2}$$

In some cases, as in the problem at hand, the designer has the freedom to choose $\mathbf{p}_0$; when this is the case, $\mathbf{p}_0$ can be chosen so as to yield an optimally robust design, its components contributing to those of the DV vector $\mathbf{x}$. Given the randomness of the DEP, we also need models for their probability distributions. For the sake of conciseness, we assume that the variations of the DEP obey Gaussian distributions with nonzero mean and nonidentical standard deviations. One advantage of the Gaussian distribution lies in its simplicity, for it is fully described by two parameters per random variable, the mean and the standard deviation. Of course, simplicity per se is of little use to real-world problems. It turns out, however, that the Gaussian distribution reflects faithfully the actual distribution found in practice. Indeed, the justification for representing many complicated phenomena by Gaussian density functions lies in the *Central Limit Theorem* [17]. Furthermore, if the mean and the covariance matrix of $\Delta\mathbf{p}$ are denoted by $\boldsymbol{\mu}_p$ and $\mathbf{P}$, respectively, then, the corresponding mean $\boldsymbol{\mu}_f$ and the covariance matrix $\boldsymbol{\Phi}$ of $\Delta\mathbf{f}$ are given by

$$\boldsymbol{\mu}_f = \mathbf{F}\boldsymbol{\mu}_p\,, \quad \boldsymbol{\Phi} = \mathbf{F}\mathbf{P}\mathbf{F}^T \tag{3}$$

For a robust design, we aim at minimizing an objective function derived from the covariance $\boldsymbol{\Phi}$ of $\Delta\mathbf{f}$. As $\boldsymbol{\Phi}$ is a matrix, any matrix norm, labeled σ_f, can be adopted, i.e.,

$$\sigma_f = \|\boldsymbol{\Phi}\| = \|\mathbf{F}\mathbf{P}\mathbf{F}^T\| \tag{4}$$

where $\|\cdot\|$ indicates *any* norm of its matrix argument $(\cdot)$.

2.3 The Robust Design As An Optimization Problem

For the purpose of achieving a robust design, we want $\|\boldsymbol{\Phi}\|$ of Eqn. (4) to be a minimum. The robust design problem is thus of a minimum-covariance type. Formally, this can be stated as a matrix-norm minimization problem of the form

$$\sigma_f \equiv \|\mathbf{F}\mathbf{P}\mathbf{F}^T\| \longrightarrow \min_{\mathbf{x}} \tag{5a}$$

subject to

$$\mathbf{f}(\mathbf{x};\mathbf{p}_0) = \mathbf{f}_0 \tag{5b}$$

In some cases, as in the problem formulated here, $\mathbf{p}_0$ is one more (vector) variable to be determined by the designer as an outcome of the design task at hand. To solve the optimization problem (5a & b), we need first to decide on the matrix norm to be adopted, which should be consistent with the Euclidean vector norm used throughout the paper. The Frobenius norm [18] is the most attractive for our needs; this norm can be defined for any $m \times n$ matrix $\mathbf{A}$ as

$$\|\mathbf{A}\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2} = \sqrt{\mathrm{tr}(\mathbf{A}\mathbf{A}^T)} \tag{6}$$

where a_{ij} is the (i,j) entry of $\mathbf{A}$, and $\mathrm{tr}(\cdot)$ is the trace of its square-matrix argument. The matrix Frobenius norm offers a computational advantage over its Euclidean counterpart, as it does not require any singular-value computations [18]. Moreover, the matrix Frobenius norm is not only computationally simple; it is also an *analytic* function of its matrix argument, which makes its minimization tractable with *gradient methods*. Indeed, the Frobenius norm allows a straightforward derivation of expressions of gradients of the objective function with respect to the design variables. If this norm is adopted, then the expression for σ_f of Eqn. (4) reduces to

$$\sigma_f \equiv \|\mathbf{F}\mathbf{P}\mathbf{F}^T\|_F = \sqrt{\mathrm{tr}[(\mathbf{F}\mathbf{P}\mathbf{F}^T)^2]} = \sqrt{\mathrm{tr}[(\mathbf{F}^T\mathbf{F}\mathbf{P})^2]} \tag{7}$$

or

$$\sigma_f^2 = \mathrm{tr}[(\mathbf{F}^T\mathbf{F}\mathbf{P})^2] \tag{8}$$

where a property of the trace of a product of matrices has been recalled: $\mathrm{tr}(\mathbf{ABC}) = \mathrm{tr}(\mathbf{CAB}) = \mathrm{tr}(\mathbf{BCA})$, for matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ compatible under multiplication. If we let $\mathbf{M} = \mathbf{F}^T\mathbf{F}\mathbf{P}$ for simplicity, the partial derivative of σ_f^2 with respect to the ith DV x_i can be readily computed:

$$\frac{\partial \sigma_f^2}{\partial x_i} = 2\,\mathrm{tr}\left(\mathbf{M}\frac{\partial \mathbf{M}^T}{\partial x_i}\right) \tag{9}$$

which requires only the derivatives of $\mathbf{M}$ with respect to the DV.

Notice, however, that minimizing σ_f as given above requires knowing $\mathbf{P}$, which is not always available. In the absence of knowledge of $\mathbf{P}$, we can instead minimize an *upper bound* of σ_f, obtained from Eqn. (8) upon invoking a matrix-norm inequality [18], namely,

$$\sigma_f \le \|\mathbf{F}\|_F\|\mathbf{P}\|_F\|\mathbf{F}^T\|_F = \|\mathbf{F}\|_F^2\|\mathbf{P}\|_F \tag{10}$$

Consequently, the general robust design problem can be formulated as one of the minimization of the bound given by Eqn. (10), which comprises two factors, $\|\mathbf{F}\|_F^2$ and $\|\mathbf{P}\|_F$. The designer not having any control over the second factor, the minimization problem at hand reduces to minimizing the first of the two factors. That is,

$$z(\mathbf{x}) \equiv \|\mathbf{F}\|_F^2 = \mathrm{tr}(\mathbf{F}\mathbf{F}^T) \to \min_{\mathbf{x}} \tag{11a}$$

subject to

$$\mathbf{f}(\mathbf{x};\mathbf{p}_0) = \mathbf{f}_0 \tag{11b}$$

Consequently, the robust design problem can be formally stated as a constrained optimization problem, which is classical in the realm of design optimization, and can be solved numerically by a host of methods [19].

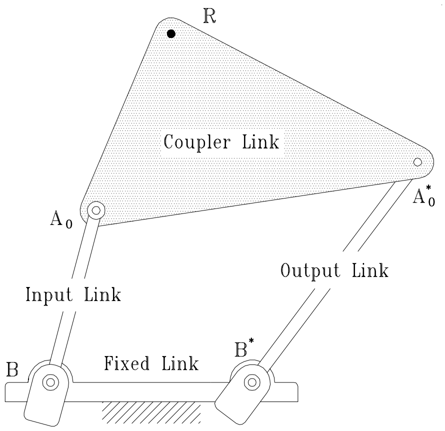

FIGURE 1. THE FOUR-BAR LINKAGE DESCRIPTION

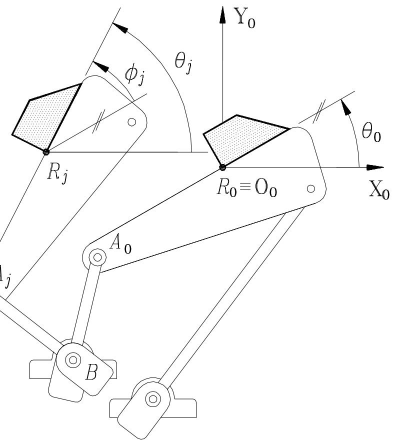

FIGURE 2. THE *j*th POSE OF A FOUR-BAR LINKAGE

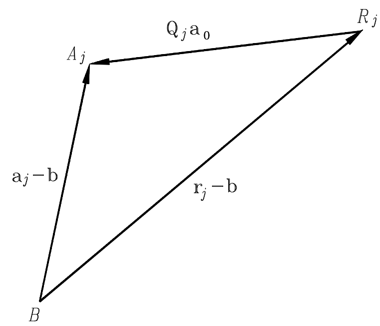

FIGURE 3. VECTORS AT THE *j*th POSE

Having formulated the robust design problem in general, we focus now on the synthesis problem of planar rigid-body guidance. We aim at finding exact four-bar motion generators able to visit a set of given poses, in addition to being robust against variations in the intermediate ones. Within the framework proposed here, the latter become part of the design variables, in addition to the linkage dimensions. Moreover, the performance functions, in our case, are the synthesis equations associated with the linkage at hand. Below we recall these synthesis equations and formulate the corresponding problem for the planar linkage synthesis.

3 THE ROBUST PLANAR LINKAGE SYNTHESIS

The planar four-bar linkage under design is depicted in Fig 1. This linkage is meant to carry the body rigidly attached to its coupler link through a set of $N+1$ prescribed poses, which are given with respect to a certain reference pose as $\{\mathbf{r}_j, \theta_j\}_0^N$, $\mathbf{r}_j$ being the position vector of a point R fixed to the coupler link at the j-th pose, while θ_j is the angle defining the j-th orientation of the guided body, as shown in Fig 2.

We recall [1] that for N prescribed poses, the number of solution linkages to visit these poses depends on N itself: For a single given pose, there is no guidance involved. For $N = 1$, the case is typical of pick-and-place operations. In order to fully determine this kind of task, additional conditions, on velocity and acceleration, are imposed to guarantee that the body is both picked and placed at rest with respect to another body, which can be, in turn, either fixed or moving. The case of $N = 2$ allows the arbitrary choice of either A_0 and A_0^* or B and B^*, the problem then becoming linear and leading to one unique solution. For $N = 3$, the problem leads to a system of three cubic equations in four unknowns; infinitely-many solutions are available, these solutions defining two related loci: the centerpoint and the circlepoint (cubic) curves. The case of $N = 4$ leads to a system of four algebraic equations in four unknowns. This problem is thus determined, but nonlinear. With $N > 4$, the problem is overdetermined: no solution is possible, but suitable approximations can be found by means of optimization techniques. Here we focus on exact synthesis, with $N < N_{\max} = 4$, where we have an incomplete set of poses, i. e., $N = 1, 2$, or 3.

3.1 The Kinematic Synthesis Equation

The jth performance function, in our case, is the jth synthesis equation defining the classic Burmester problem [1]. Without loss of generality, let us define the origin of the fixed coordinate frame in such a way that $\mathbf{r}_0 \equiv \mathbf{0}_2 = [0\ 0]$, as shown in Fig. 2. Moreover, by defining relative angles $\phi_j \equiv \theta_j - \theta_0$, the set of given poses reduces to $\{\mathbf{r}_j, \phi_j\}_1^N$. Shown in Fig. 3 is a sketch of the vectors defining an arbitrary configuration of the four-bar linkage.

Referring to Fig. 3, the condition under which the distance between points A_0 and B—representing a link length—remains constant yields N constraint equations [15], namely,

$$\|\mathbf{r}_j - \mathbf{b} + \mathbf{Q}_j \mathbf{a}_0\| = \|\mathbf{a}_0 - \mathbf{b}\|, \quad j = 1, \ldots, N \tag{12a}$$

where $\mathbf{b}$ is the position vector of point B, and $\mathbf{a}_0$ that of point A_0. Moreover, $\mathbf{Q}_j$ is the matrix that rotates a vector from the 0-th pose to the j-th pose, and is given by

$$\mathbf{Q}_j = \begin{bmatrix} c\phi_j & -s\phi_j \\ s\phi_j & c\phi_j \end{bmatrix} = c\phi_j \mathbf{1}_2 + s\phi_j \mathbf{E}, \quad \mathbf{E} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{12b}$$

where $c\phi_j$ and $s\phi_j$ stand for $\cos\phi_j$ and $\sin\phi_j$, respectively, $\mathbf{1}_2$ is the 2×2 identity matrix, and $\mathbf{E}$ is the matrix rotating a vector through an angle of $90°$ counterclockwise. After squaring and expanding both sides of Eqn. (12a), and then simplifying, we obtain the synthesis equations:

$$(\mathbf{r}_j - \mathbf{b})^T \mathbf{Q}_j \mathbf{a}_0 + \mathbf{b}^T \mathbf{a}_0 + \mathbf{r}_j^T (\frac{\mathbf{r}_j}{2} - \mathbf{b}) = 0, \;\; j = 1, \ldots, N \tag{13}$$

which can be further expanded using Eqn. (12b) to yield

$$f_j \equiv (\mathbf{b} - \mathbf{r}_j)^T \mathbf{a}_0 c\phi_j + (\mathbf{b} - \mathbf{r}_j)^T \mathbf{E} \mathbf{a}_0 s\phi_j + (\mathbf{a}_0 - \mathbf{r}_j)^T \mathbf{b} + \frac{1}{2}\|\mathbf{r}_j\|^2 = 0, \quad j = 1, \ldots, N \tag{14a}$$

By the same token, we can write the jth synthesis equation for the $B^* A_0^*$-dyad of Fig. 1 as

$$f_j \equiv (\mathbf{b}^* - \mathbf{r}_j)^T \mathbf{a}_0^* c\phi_j + (\mathbf{b}^* - \mathbf{r}_j)^T \mathbf{E} \mathbf{a}_0^* s\phi_j + (\mathbf{a}_0^* - \mathbf{r}_j)^T \mathbf{b}^* + \frac{1}{2}\|\mathbf{r}_j\|^2 = 0, \quad j = N+1, \ldots, 2N \tag{14b}$$

3.2 The Robust Kinematic Synthesis Formulation

In our case, the k unspecified poses $\{\phi_j, \mathbf{r}_j\}_{4-k}^4$ represent the DEP, hence, the $3k$-dimensional DEP vector is

$$\mathbf{p} \equiv \begin{bmatrix} \boldsymbol{\pi}_4 \\ \vdots \\ \boldsymbol{\pi}_{4-k} \end{bmatrix}, \quad \boldsymbol{\pi}_j = \begin{bmatrix} \phi_j \\ \mathbf{r}_j \end{bmatrix}, \quad j = 4-k, \ldots, 4 \tag{15}$$

while its nominal value $\mathbf{p}_0$, as yet to be determined, has to be considered as part of the design variables. Moreover, within the framework proposed here, the design variables are both the linkage parameters $\{\mathbf{a}_0, \mathbf{b}, \mathbf{a}_0^*, \mathbf{b}^*\}$ and the k unspecified poses $\{\phi_j, \mathbf{r}_j\}_{4-k}^4$, i.e.,

$$\mathbf{x} \equiv [\, \mathbf{a}_0^T \;\; \mathbf{b}^T \;\; \mathbf{a}_0^{*T} \;\; \mathbf{b}^{*T} \;\; \phi_{4-k} \;\; \mathbf{r}_{4-k}^T \; \ldots \; \phi_4 \;\; \mathbf{r}_4^T \,]^T \tag{16}$$

Apparently, we have $8 + 3k$ design variables to determine. Furthermore, the PFV is

$$\mathbf{f} \equiv \left[\, f_{4-k} \ldots f_4 \;\; f_{8-k} \; \ldots \; f_8 \,\right]^T \tag{17}$$

where we have excluded $f_1, \ldots, f_{3-k}, f_5, \ldots, f_{7-k}$ because these functions are independent of $\mathbf{p}$. However, their vanishing will be added to the constraints.

Resorting to Eqns. (14a & b), Δf_j can be obtained in terms of nondimensional variations $\Delta\phi_j$ and $\Delta\mathbf{r}_j / \|\mathbf{r}_j\|$, for $\|\mathbf{r}_j\| \neq 0$, as[1]

$$\Delta f_j = \left(\frac{\partial f_j}{\partial \phi_j}\right) \Delta\phi_j + \|\mathbf{r}_j\| \left(\frac{\partial f_j}{\partial \mathbf{r}_j}\right)^T \frac{\Delta\mathbf{r}_j}{\|\mathbf{r}_j\|} \tag{18}$$

The above relations can be cast in the form

$$\Delta f_j = \left(\frac{\partial f_j}{\partial \widetilde{\mathbf{p}}_j}\right)^T \Delta\widetilde{\mathbf{p}}_j, \quad j = 4-k, \ldots, 4, 8-k, \ldots, 8 \tag{19}$$

in which

$$\frac{\partial f_j}{\partial \widetilde{\mathbf{p}}_j} = \begin{bmatrix} \dfrac{\partial f_j}{\partial \phi_j} \\ \\ \|\mathbf{r}_j\| \dfrac{\partial f_j}{\partial \mathbf{r}_j} \end{bmatrix}, \quad \Delta\widetilde{\mathbf{p}}_j = \begin{bmatrix} \Delta\phi_j \\ \Delta\widetilde{\mathbf{r}}_j \end{bmatrix}, \quad \Delta\widetilde{\mathbf{r}}_j = \frac{\Delta\mathbf{r}_j}{\|\mathbf{r}_j\|} \tag{20}$$

The $2k \times 3k$ performance matrix $\mathbf{F}$ relating changes in the performance functions upon changes in the design-environment parameters takes the form

$$\mathbf{F} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \tag{21a}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are $k \times 3k$ block-diagonal matrices of the form

$$\mathbf{A}_1 = \begin{bmatrix} \left(\dfrac{\partial f_{4-k}}{\partial \widetilde{\mathbf{p}}_{4-k}}\right)^T & \mathbf{0}_3^T & \cdots & \mathbf{0}_3^T \\ \mathbf{0}_3^T & \left(\dfrac{\partial f_{5-k}}{\partial \widetilde{\mathbf{p}}_{5-k}}\right)^T & \cdots & \mathbf{0}_3^T \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_3^T & \mathbf{0}_3^T & \cdots & \left(\dfrac{\partial f_4}{\partial \widetilde{\mathbf{p}}_4}\right)^T \end{bmatrix} \tag{21b}$$

[1] Unless all given poses represent rotations about the same point R_0, a rather unlikely case, it is always possible to define R_0 such as $\|\mathbf{r}_j\| \neq \mathbf{0}$, for $j = 0, \ldots, N$.

and

$$\mathbf{A}_2 = \begin{bmatrix} \left(\frac{\partial f_{8-k}}{\partial \widetilde{\mathbf{p}}_{8-k}}\right)^T & \mathbf{0}_3^T & \cdots & \mathbf{0}_3^T \\ \mathbf{0}_3^T & \left(\frac{\partial f_{9-k}}{\partial \widetilde{\mathbf{p}}_{9-k}}\right)^T & \cdots & \mathbf{0}_3^T \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_3^T & \mathbf{0}_3^T & \cdots & \left(\frac{\partial f_8}{\partial \widetilde{\mathbf{p}}_8}\right)^T \end{bmatrix} \tag{21c}$$

with $\mathbf{0}_3$ representing the three-dimensional zero vector. It is noteworthy that the foregoing partial derivatives are taken with respect to the uncertain intermediate poses, then evaluated at $\mathbf{p}_0$. Now, the partial derivatives involved are readily computed resorting to Eqns. (14a & b), namely,

$$\frac{\partial f_j}{\partial \phi_j} = \begin{cases} -s\phi_j(\mathbf{b}-\mathbf{r}_j)^T\mathbf{a}_0 + c\phi_j(\mathbf{b}-\mathbf{r}_j)^T\mathbf{E}\mathbf{a}_0, & j = 4-k,\ldots,4 \\ -s\phi_j(\mathbf{b}^*-\mathbf{r})^T\mathbf{a}_0^* + c\phi_j(\mathbf{b}^*-\mathbf{r}^*)^T\mathbf{E}\mathbf{a}_0^*, & j = 8-k,\ldots,8 \end{cases}$$

and

$$\frac{\partial f_j}{\partial \mathbf{r}_j} = \begin{cases} -c\phi_j\mathbf{a}_0 - s\phi_j\mathbf{E}\mathbf{a}_0 - \mathbf{b} + \mathbf{r}_j, & j = 4-k,\ldots,4 \\ -c\phi_j\mathbf{a}_0^* - s\phi_j\mathbf{E}\mathbf{a}_0^* - \mathbf{b}^* + \mathbf{r}_j, & j = 8-k,\ldots,8 \end{cases}$$

Hence, the cost function $z(\mathbf{x})$ of the optimization problem (11) is evaluated as

$$z(\mathbf{x}) = \sum_{i=1}^{2} \|\mathbf{A}_i\|_F^2 = \sum_{j=4-k}^{4} \left\|\frac{\partial f_j}{\partial \widetilde{\mathbf{p}}_j}\right\|^2 + \sum_{j=8-k}^{8} \left\|\frac{\partial f_j}{\partial \widetilde{\mathbf{p}}_j}\right\|^2 \tag{22}$$

which can be further reduced to

$$z(\mathbf{x}) = \sum_{j=4-k}^{5} \left\{ \left(\frac{\partial f_j}{\partial \phi_j}\right)^2 + \|\mathbf{r}_j\|^2 \left\|\frac{\partial f_j}{\partial \mathbf{r}_j}\right\|^2 \right\} + \sum_{j=8-k}^{8} \left\{ \left(\frac{\partial f_j}{\partial \phi_j}\right)^2 + \|\mathbf{r}_j\|^2 \left\|\frac{\partial f_j}{\partial \mathbf{r}_j}\right\|^2 \right\} \tag{23}$$

Therefore, the synthesis problem is formulated as

$$z(\mathbf{x}) = \sum_{j=4-k}^{4} \left\{ \left(\frac{\partial f_j}{\partial \phi_j}\right)^2 + \|\mathbf{r}_j\|^2 \left\|\frac{\partial f_j}{\partial \mathbf{r}_j}\right\|^2 \right\} + \sum_{j=8-k}^{8} \left\{ \left(\frac{\partial f_j}{\partial \phi_j}\right)^2 + \|\mathbf{r}_j\|^2 \left\|\frac{\partial f_j}{\partial \mathbf{r}_j}\right\|^2 \right\} \longrightarrow \min_{\mathbf{x}} \tag{24a}$$

subject to

$$f_j = \begin{cases} c\phi_j(\mathbf{b}-\mathbf{r}_j)^T\mathbf{a}_0 + s\phi_j(\mathbf{b}-\mathbf{r}_j)^T\mathbf{E}\mathbf{a}_0 \\ +(\mathbf{a}_0-\mathbf{r}_j)^T\mathbf{b} + \frac{1}{2}\|\mathbf{r}_j\|^2 = 0, \quad j = 1,\ldots,N \\ c\phi_j(\mathbf{b}^*-\mathbf{r}_j)^T\mathbf{a}_0^* + s\phi_j(\mathbf{b}^*-\mathbf{r}_j)^T\mathbf{E}\mathbf{a}_0^* \\ +(\mathbf{a}_0^*-\mathbf{r}_j)^T\mathbf{b}^* + \frac{1}{2}\|\mathbf{r}_j\|^2 = 0, \ j = N+1,\ldots,2N \end{cases} \tag{24b}$$

4 A Numerical Example

Here we consider the robust synthesis of a planar four-bar linkage, as shown in Fig. 2, for pick-and-place operations, in which the linkage is meant to take a rigid body, attached to its coupler link, from a given *initial pose*, specified by the position of one of its points and its orientation with respect to a given coordinate frame, to a *final pose*, specified likewise, with the same landmark point. However, how the rigid body moves from its initial to its final pose is left unspecified, with the only condition that no collisions are produced either between links or between the linkage and other objects in its surroundings. Let us consider the operation of taking a rigid body from a "pick" pose to a "place" pose reached by rotating and translating the rigid body, as depicted in Fig. 4, through

$$\mathbf{r}_4 = \begin{bmatrix} 2.3000 \\ 3.5000 \end{bmatrix}, \quad \phi_4 = 90^\circ$$

The built-in function `constr` of MATLAB's Optimization Toolbox [20] is used here for finding numerical solutions of the optimization problem under study. This function uses a Sequential Quadratic Programming (SQP) method. In this case, a solution of the three intermediate poses $\mathbf{p}_0$ turned out to be

$$\mathbf{p}_0 = \begin{bmatrix} \boldsymbol{\pi}_3 \\ \boldsymbol{\pi}_2 \\ \boldsymbol{\pi}_1 \end{bmatrix}$$

with

$$\boldsymbol{\pi}_1 = \begin{bmatrix} -10.4578^\circ \\ 1.3999 \\ 0.8320 \end{bmatrix}, \ \boldsymbol{\pi}_2 = \begin{bmatrix} 19.2378^\circ \\ 1.7678 \\ 1.6011 \end{bmatrix}, \ \boldsymbol{\pi}_3 = \begin{bmatrix} 60.9990^\circ \\ 2.0010 \\ 3.1999 \end{bmatrix}$$

The corresponding linkage vectors are

$$\mathbf{a}_0 = \begin{bmatrix} 0.2376 \\ 2.8657 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0.3423 \\ 3.767 \end{bmatrix}, \mathbf{a}_0^* = \begin{bmatrix} 1.9234 \\ 2.9847 \end{bmatrix}, \mathbf{b}^* = \begin{bmatrix} 0.9942 \\ 4.5747 \end{bmatrix}$$

A linkage capable of producing the given motion is shown in Fig. 4. Hence, The link lengths can be obtained as

$$\overline{BA_0} = 0.9074, \ \overline{A_0A_0^*} = 1.6700, \ \overline{B^*A_0^*} = 1.846, \ \overline{BB^*} = 1.0380$$

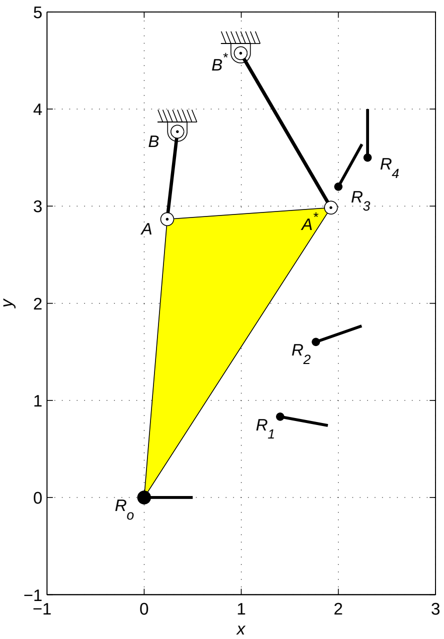

FIGURE 4. SOLUTION LINKAGE FOR THE EXAMPLE

5 CONCLUSIONS

A methodology for the robust synthesis of planar four-bar linkages to guide their coupler link through an incomplete set of prescribed poses was proposed in this paper. Robustness is intended to minimize variations in the linkage performance against variations in the unspecified poses. Robustness is considered as a design criterion for selecting the unprescribed intermediate poses. Moreover, these poses, regarded as the design environment parameters, are chosen so as to minimize the sensitivity of the linkage performance to changes in these poses.

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