Chapter 30: Finite-Difference: Parabolic Equation

parabolic (Heat) equation

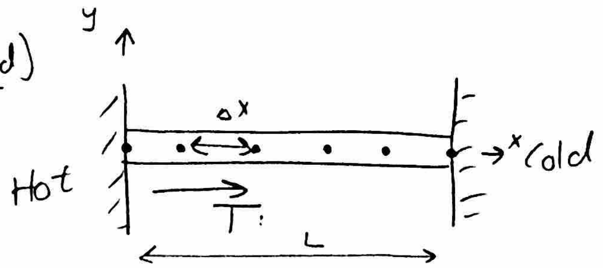
$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

T: Temperature $T(x,t)$
t: Time
x: space (location)
k: constant.

* Finite Difference method (Explicit method)

2nd derivative using centered Finite difference approximation

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}$$



Divide $x \rightarrow \Delta x$
Divide $t \rightarrow \Delta t$ $(t_0 \rightarrow t)$

and 1st derivative using Forward finite difference approximations

$$\frac{\partial T}{\partial t} = \frac{T_{i,j+1} - T_{i,j}}{\Delta t}$$

Δx: step size in x
Δt: step size in t
i: space counter
j: time counter

substitute in PDE and let $\lambda = k\Delta t/(\Delta x)^2$

$$\boxed{T_{i,j+1} = T_{i,j} + \lambda\left(T_{i+1,j} - 2T_{i,j} + T_{i-1,j}\right)}$$

Heat difference equation

$i, j = 0, 1, 2, \ldots, n$

To solve heat equation, we need 2 boundary conditions and one initial condition.

BC's

$T(0,t) = T_1$
$T(L,t) = T_2$

, IC's

$T(x,0) = f(x)$

Example : For a bar with 10 cm length, the heat equation is:

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}$$

BC's
$T(0,t) = 100°C$
$T(10,t) = 50°C$

IC's
$T(x,0) = 0$

Use Explicit method to solve this PDE. Given $\Delta x = 2$ cm
$\Delta t = 0.1$ and $K = 0.49$. Total time t $0 \rightarrow 0.2$ seconds

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Solution

i	x_i
0	$x_0 = 0$
1	$x_1 = 2$
2	$x_2 = 4$
3	$x_3 = 6$
4	$x_4 = 8$
5	$x_5 = 10$

j	t_j
0	$t_0 = 0$
1	$t_1 = 0.1$
2	$t_2 = 0.2$

(*) we know ① at $x_0 = 0$, $T = 100°C$ for all t

② at $x_5 = 10$, $T = 50°C$ for all t

③ x_i is always zero for $t_0 = 0$

$\Rightarrow$ we need

x_1, t_1
x_2, t_1
x_3, t_1
x_4, t_1

x_1, t_2
x_2, t_2
x_3, t_2
x_4, t_2

Now, use

$$T_{i,j+1} = T_{i,j} + \lambda\left(T_{i+1,j} - 2T_{i,j} + T_{i-1,j}\right) \quad , \quad \lambda = \frac{k\Delta t}{(\Delta x)^2} = 0.02$$

For x_1, t_1 $(i=1, j=0)$

$$T_{1,1} = T_{10} + \lambda\left(T_{2,0} - 2T_{1,0} + T_{0,0}\right)$$

$$T_{10} = 0, \quad T_{2,0} = 0, \quad T_{0,0} = 100$$

$$\Rightarrow T_{1,1} = 0 + 0.02(0 - (2)(0) - 100) \Rightarrow \boxed{T_{1,1} = 2\ ^\circ C}$$

For point x_2, t_1 $(i=2, j=0)$

$$T_{2,1} = T_{2,0} + \lambda\left(T_{3,0} - 2T_{2,0} + T_{1,0}\right)$$

$$T_{2,0} = 0, \quad T_{3,0} = 0, \quad T_{1,0} = 0$$

$$\Rightarrow \boxed{T_{2,1} = 0}$$

For point x_3, t_1 $(i=3, j=0)$

$$T_{3,1} = T_{3,0} + \lambda\left(T_{4,0} - 2T_{3,0} + T_{2,0}\right)$$

$$T_{3,0} = 0, \quad T_{4,0} = 0, \quad T_{2,0} = 0$$

$$\Rightarrow T_{3,1} = 0$$

Keep going to compute T at all other points

See example 30.1

Boundary conditions

1- Fixed type (Dirichlet) → we discussed this in Previous example
2- Derivative type (Neumann)

● Derivative type or Neumann BC's

For the previous example, If $\frac{dT}{dx}(0,t) = f(t)$

- To Find temperature at $x_0 = 0$ and t_j $(i=0, j=j)$

$$T_{0,j+1} = T_{0,j} + \lambda\left(T_{1,j} - 2T_{0,j} + T_{-1,j}\right)$$

For the derivative BC

$$\frac{dT}{dx} = \frac{T_{1,j} - T_{-1,j}}{2\Delta x} \Rightarrow T_{-1,j} = T_{1,j} - 2\Delta x \frac{dT}{dx}$$

⇒ At $x_0 = 0$, any t

$$T_{0,j+1} = T_{0,j} + \lambda\left(T_{1,j} - 2T_{0,j} + T_{1,j} - 2\Delta x \frac{dT}{dx}\right)$$

$$T_{0,j+1} = T_{0,j} + 2\lambda\left(T_{1,j} - T_{0,j} - \Delta x\, f(t_j)\right)$$

Solve problem 30.2 For more details

Chapter 30, Heat - Equation, cont'd

30.3 A simple Implicit method

In this method, we find the find differences at t_{j+1}, thus:

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,\,j+1} - 2T_{i,\,j+1} + T_{i-1,\,j+1}}{(\Delta x)^2}$$

$$\frac{\partial T}{\partial t} = \frac{T_{i,j+1} - T_{i,j}}{\Delta t}$$

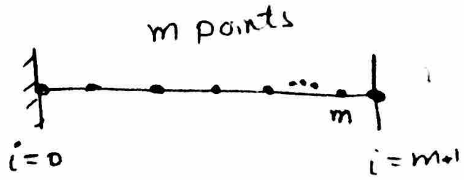
Substitute in PDE $\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}$

$$\Rightarrow -\lambda\, T_{i-1,\,j+1} + (1+2\lambda)\, T_{i,j+1} - \lambda\, T_{i+1,\,j+1} = T_{i,j}$$

where $\lambda = \frac{K\,\Delta t}{(\Delta x)^2}$

This equation applies to all points but not boundary points (start and end points) $\Rightarrow$ Modification required!

For left point $(i=0)$



$T_{0,\,j+1} = f_0(t_{j+1})$, the BC

For point $i=1$ and any t

$$(1+2\lambda)\, T_{1,\,j+1} - \lambda\, T_{2,\,j+1} = T_{1,j} + \lambda\, f_0(t_{j+1})$$

$\vdots$

For point $(i=m)$, before the last end.

$$-\lambda\, T_{m-1,\,j+1} + (1+2\lambda)\, T_{m,\,j+1} = T_{m,j} + \lambda\, f_{m+1}(t_{j+1})$$

the BC

$f_{m+1}(t_{j+1}) = T_{m+1,\,j+1}$

Example For the previous example, use the Implicit method!

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2},$$

$L = 10$ cm

BC's

$T(0,t) = 100^\circ C$

$T(10,t) = 50^\circ C$

IC's

$T(x,0) = 0$

$\Delta x = 2$ cm, $\Delta t = 0.1$, total time 0.2 seconds, $k = 0.49$

Solution $\lambda = \frac{k\,\Delta t}{(\Delta x)^2} = 0.02$

i	x_i
0	$x_0 = 0$
1	$x_1 = 2$
2	$x_2 = 4$
3	$x_3 = 6$
4	$x_4 = 8$
5	$x_5 = 10$

j	t_j
0	$t_0 = 0$
1	$t_1 = 0.1$
2	$t_2 = 0.2$

we need to Find $x_1 \rightarrow x_4$ for t_1 and t_2

① at t_1 (take $j=0$)

point x_1 $(i=1)$

$$(1+2\lambda)T_{1,j+1} - \lambda T_{2,j+1} = T_{1,j} + \lambda f_0(t_{j+1})$$

$$(1+(2)(0.02))T_{1,1} - 0.02\,T_{2,1} = \overbrace{T_{1,0}}^{=0} + \lambda(100)$$

$$\Rightarrow \quad 1.04\,T_{1,1} - 0.02\,T_{2,1} = 2.0 \qquad \text{— Eq(1)}$$

For point $x_2 \Rightarrow (i=2)$

$$-0.02\,T_{1,1} + (1+(2)(0.02))T_{2,1} - 0.02\,T_{3,1} = \overbrace{T_{2,0}}^{=0}$$

$$\Rightarrow \quad -0.02\,T_{1,1} + 1.04\,T_{2,1} - 0.02\,T_{3,1} = 0 \qquad \text{— Eq(2)}$$

For point $x_3 \Rightarrow i=3$

$$-0.02\,T_{2,1} + (1+(2)(0.02))\,T_{3,1} - 0.02\,T_{4,1} = \underbrace{T_{3,0}}^{=0}$$

$$\Rightarrow \quad -0.02\,T_{2,1} + 1.04\,T_{3,1} - 0.02\,T_{4,1} = 0 \quad \text{— Eq (3)}$$

For point x_4 $(i = 4 = m)$

BC

$$-0.02\,T_{3,1} + (1+(2)(0.02))\,T_{4,1} = \overbrace{T_{4,0}}^{=0} + 0.02\,\overbrace{f_5(t_{j+1})}^{=50^\circ C}$$

$$-0.02\,T_{3,1} + 1.04\,T_{4,1} = 1 \quad \text{— Eq(4)}$$

Matrix Form

$$\begin{bmatrix} 1.04 & -0.02 & 0 & 0 \\ -0.02 & 1.04 & -0.02 & 0 \\ 0 & -0.02 & 1.04 & -0.02 \\ 0 & 0 & -0.02 & 1.04 \end{bmatrix} \begin{Bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{4,1} \end{Bmatrix} = \begin{Bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

Solve $\Rightarrow$ $T_{1,1} = 2.00$, $T_{2,1} = 0.041$, $T_{3,1} = 0.021$, $T_{4,1} = 1.00$

For $t_2 = 0.02$ use $j=1$ and repeat all pant $x_1 \to x_4$ to get another system of 4 - linear equations

* The coefficient matrix, remains the same. However, $\{T_{i,2}\}$ and resultant vectors should change

Solve $\Rightarrow$ $T_{1,2} = 3.93$

$T_{2,2} = 0.12$

$T_{3,2} = 0.07$

$T_{4,2} = 1.96$

30.4 Crank – Nicolson method (CN–method)

* CN method is another Implicit method used to solve heat equation

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

* However, it combines both Explicit and implicit solutions.

$$\frac{\partial T}{\partial t} = \frac{T_{i,j+1} - T_{i,j}}{\Delta t}$$

and

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{2}\left[\underbrace{\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}}_{\text{Explicit}} + \underbrace{\frac{T_{i+1,j+1} - 2T_{i,j+1} + T_{i-1,j+1}}{(\Delta x)^2}}_{\text{Implicit}}\right]$$

Averages both explicit and Implicit methods!

Substitute in the PDE and collect terms: Eq 1-C

$$-\lambda T_{i-1,j+1} + 2(1+\lambda) T_{i,j+1} - \lambda T_{i+1,j+1} = \lambda T_{i-1,j} + 2(1-\lambda)T_{i,j} + \lambda T_{i+1,j}$$

where $\lambda = \frac{k\,\Delta t}{(\Delta x)^2}$

As we did for the implicit method, at the boundary condition point ($i=0$) $T_{0,j+1} = f_0(t_{j+1})$ and at the ($i=m$), point $T_{m+1,j+1} = f_{m+1}(t_{j+1})$

Thus, for point ($i=1$) at any t. Eq2-C

$$2(1+\lambda) T_{1,j+1} - \lambda T_{2,j+1} = \underbrace{\lambda f_0(t_j)}_{BC} + 2(1-\lambda) T_{1,j} + \lambda f_0(t_{j+1})$$

For point ($i=m$), before the last end point Eq3-C

$$-\lambda T_{m-1,j+1} + 2(1+\lambda) T_{m,j+1} = \lambda f_{m+1}(t_j) + 2(1-\lambda) T_{m,j} + \lambda T_{m-1,j} + \lambda f_{m+1}(t_{j+1})$$

Example8- For the previous example, use CN method to solve

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}, \quad L = 10cm, \quad \text{BC's: } T(0,t) = 100°C, \; T(10,t) = 50°C \qquad \text{IC's: } T(x,0) = 0$$

$\Delta x = 2\,cm$, $\Delta t = 0.1$ sec, total time 0.2 seconds, $k = 0.49$

Solution $\lambda = \frac{k\,\Delta t}{(\Delta x)^2} = 0.02$

i	x_i
0	$x_0 = 0$
1	$x_1 = 2$
2	$x_2 = 4$
3	$x_3 = 6$
4	$x_4 = 8$
5	$x_5 = 10$

j	t_j
0	$t_0 = 0$
1	$t_1 = 0.1$
2	$t_2 = 0.2$

* we need to find x_i for $i = 1 \to 4$ at t_1 and t_2

① At t_1 ($j=0$) (Eq→2-c)

point x_1 ($i=1$)

$$2(1+0.02)T_{1,1} - (0.02)T_{2,1} = (0.02)\underbrace{f_0(t_0)}_{=100 \text{ (BC)}} - (2)(1-0.02)\underbrace{T_{1,0}}_{=0 \text{ IC}} + (0.02)\underbrace{f_0(t_1)}_{=100°C \text{ (BC)}}$$

$$2.04\,T_{1,1} - 0.02\,T_{2,1} = 4.0 \quad \text{—①}$$

point x_2 ($i=2$), Eq1-C

$$-0.02\,T_{1,1} + 2(1+0.02)T_{2,1} - 0.02\,T_{3,1} = 0.02\underbrace{T_{1,0}}_{=0 \text{ IC}} + 2(1-0.02)\underbrace{T_{2,0}}_{=0 \text{ IC}} + 0.02\underbrace{T_{3,0}}_{=0 \text{ IC}}$$

$$-0.02\,T_{1,1} + 2.04\,T_{2,1} - 0.02\,T_{3,1} = 0 \quad ②$$

point x_3 ($i=3$), Eq 1-C

$$-0.02\,T_{2,1} + 2.04\,T_{3,1} - 0.02\,T_{4,1} = 0 \quad -3$$

point x_4 ($i=4$), Eq 3-C

$$-0.02\,T_{3,1} + 2.04\,T_{4,1} = 2.07 \quad -4$$

$\Rightarrow$ Collect in Matrix Form

$$\begin{bmatrix} 2.04 & -0.02 & 0 & 0 \\ -0.02 & 2.04 & -0.02 & 0 \\ 0 & -0.02 & 2.04 & -0.02 \\ 0 & 0 & -0.02 & 2.04 \end{bmatrix} \begin{Bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{4,1} \end{Bmatrix} = \begin{Bmatrix} 4.0 \\ 0 \\ 0 \\ 2.08 \end{Bmatrix}$$

Solve $T_{1,1} = 2.045$, $T_{2,1} = 0.021$, $T_{3,1} = 0.011$, $T_{4,1} = 1.023$

* For $t_2 = 0.02$ use $j=1$ and repeat for all points $x_1 \rightarrow x_4$ to get another system of 4-linear equations

* The coefficient matrix, remains the same. However, $\{T_{i,2}\}$ and resultant vectors should change

Solve $\Rightarrow$

$$T_{1,2} = 4.007$$
$$T_{2,2} = 0.083$$
$$T_{3,2} = 0.042$$
$$T_{4,2} = 2.004$$

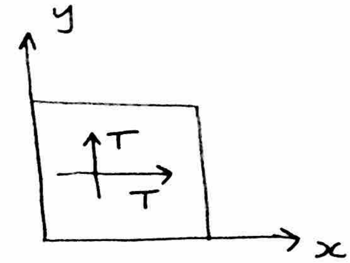
Chapter 30, Heat Equation

30.5 Heat Equation in two spatial dimensions

$$\frac{\partial T}{\partial t} = K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right),$$

$T(x,y,t)$

T: temp
x, y: space
K: constant
t: time



we use alternating-direction implicit (ADI) method. we solve the PDE for each time increment (Δt) in two steps:

1st step

$$\frac{\partial T}{\partial t} = \frac{T_{i,j}^{K+1/2} - T_{i,j}^{K}}{\Delta t/2}$$

i: x-direction counter
j: y-direction counter
K: time counter

Δx: step size in x
Δy: step size in y
Δt: step size in t

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,j}^{K} - 2T_{i,j}^{K} + T_{i-1,j}^{K}}{(\Delta x)^2}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{i,j+1}^{K+1/2} - 2T_{i,j}^{K+1/2} + T_{i,j-1}^{K+1/2}}{(\Delta y)^2}$$

Substitute in PDE, for $\Delta x = \Delta y$, $\lambda = \frac{K\,\Delta t}{(\Delta x)^2}$ or $\lambda = \frac{K\,\Delta t}{(\Delta y)^2}$

Eq(a)

$$-\lambda T_{i,j-1}^{K+1/2} + 2(1+\lambda)T_{i,j}^{K+1/2} - \lambda T_{i,j+1}^{K+1/2} = \lambda T_{i-1,j}^{K} + 2(1-\lambda)T_{i,j}^{K} + \lambda T_{i+1,j}^{K}$$

2nd step:

$$\frac{\partial T}{\partial t} = \frac{T_{i,j}^{K+1} - T_{i,j}^{K+1/2}}{\Delta t/2}, \quad \frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,j}^{K+1} - 2T_{i,j}^{K+1} + T_{i-1,j}^{K+1}}{(\Delta x)^2}, \quad \frac{\partial^2 T}{\partial y^2} = \text{as 1st step}$$

substitute in PDE, $\Delta x = \Delta y$, $\lambda = \frac{K\,\Delta t}{(\Delta x)^2}$ or $\lambda = \frac{K\,\Delta t}{(\Delta y)^2}$

Eq(b)

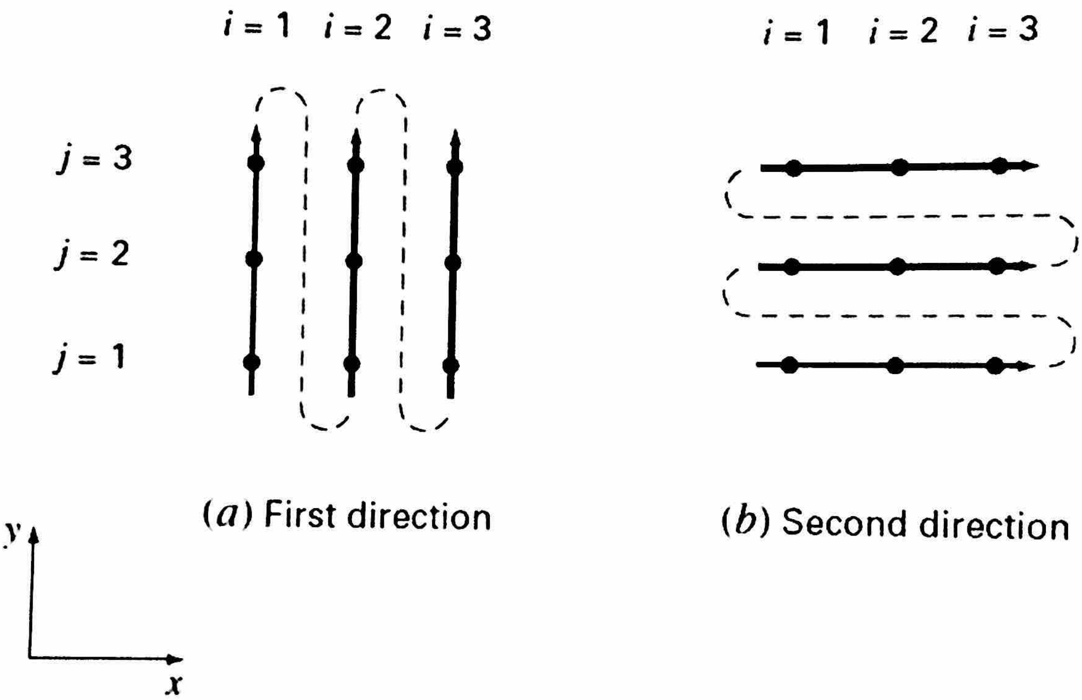
$$-\lambda T_{i-1,j}^{K+1} + 2(1+\lambda)T_{i,j}^{K+1} - \lambda T_{i,j}^{K+1} = \lambda T_{i,j-1}^{K+1/2} + 2(1-\lambda)T_{i,j}^{K+1/2} + T_{i,j+1}^{K+1/2}$$

In Eq(a), $\frac{\partial^2 T}{\partial x^2}$ are known. However, $\frac{\partial^2 T}{\partial y^2}$ are not

$\Rightarrow$ obtain all $T^{k+1/2}$ terms $\Rightarrow$ use them in Eq(b)

In Eq(b), $\frac{\partial^2 T}{\partial x^2}$ are now unknown. However, $\frac{\partial^2 T}{\partial y^2}$ are known from Eq(a)

$\Rightarrow$ obtain all T^{k+1} terms.



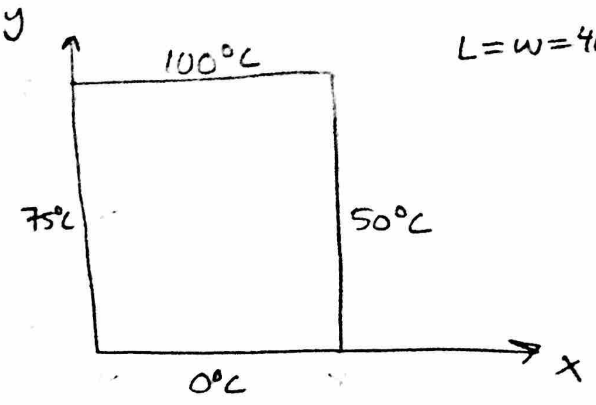
(a) First direction

(b) Second direction

we first solve for y-direction, then for x-direction at each time step

Example Use ADI, to solve

$$\frac{\partial T}{\partial t} = K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$



$L = w = 40$ cm

3

BC's

$T(0,y,t) = 75°C$

$T(L,y,t) = 50°C$

$T(x,0,t) = 0°C$

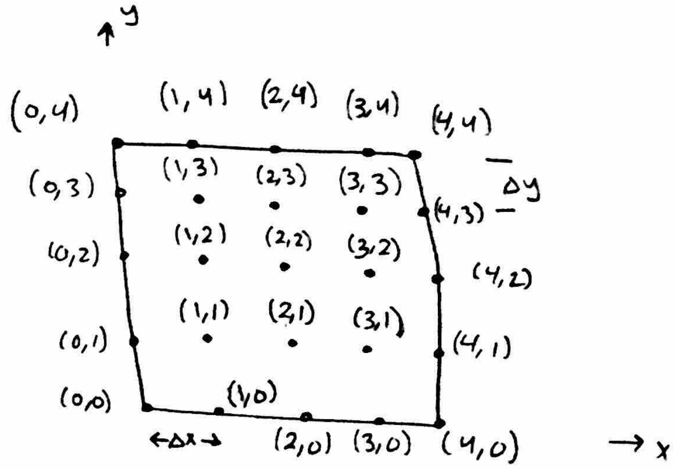
$T(x,w,t) = 100°C$

IC's

$T(x,y,0) = 0$

Use, $\Delta x = 10$ cm, $\Delta y = 10$ cm, $\Delta t = 10$ s, total time = 10 seconds, $K = 0.835$

Solution $\lambda = \frac{K\,\Delta t}{(\Delta x)^2} = \frac{(0.835)(10)}{(10)^2} = 0.0835$



we need to find $T_{i,j}$ $\left(\begin{matrix} i = 1,2,3 \text{ and} \\ j = 1,2,3 \end{matrix}\right)$

For step 1 $(t = 5$ seconds, $\Delta t/2)$, $k + \frac{1}{2}$, Eq(a)

- point (1,1)

$$\Rightarrow \quad 2.167\,T_{1,1} - 0.0835\,T_{1,2} = 6.2625$$

- point (1,2)

$$-0.0835\,T_{1,1} + 2.167\,T_{1,2} - 0.0835\,T_{1,3} = 6.2625$$

- point (1,3)

$$-0.0835\,T_{1,2} + 2.167\,T_{1,3} = 14.6125$$

Matrix Form

$$\begin{bmatrix} 2.167 & -0.0835 & 0 \\ -0.0835 & 2.167 & -0.0835 \\ 0 & -0.0835 & 2.167 \end{bmatrix} \begin{Bmatrix} T_{1,1} \\ T_{1,2} \\ T_{1,3} \end{Bmatrix} = \begin{Bmatrix} 6.2625 \\ 6.2625 \\ 14.6125 \end{Bmatrix}$$

Solve

$$T_{1,1} = 3.02, \quad T_{1,2} = 3.27, \quad T_{1,3} = 6.87$$

In a Similar fashion, we find $T_{2,1}$, $T_{2,2}$, $T_{2,3}$ and $T_{3,1}$, $T_{3,2}$, $T_{3,3}$

$T_{2,1} = 0.13$, $T_{2,2} = 3.27$, $T_{2,3} = 4.13$

$T_{3,1} = 2.02$, $T_{3,2} = 2.25$, $T_{3,3} = 6.03$

Now, we do step 2 ($t = 10$ sec , $k+1$), Eq(6)

- points (1,1), (2,1) and (3,1), matrix Form

$$\begin{bmatrix} 2.167 & -0.0835 & 0 \\ -0.0835 & 2.167 & -0.0835 \\ 0 & -0.0835 & 2.167 \end{bmatrix} \begin{Bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \end{Bmatrix} = \begin{Bmatrix} 12.064 \\ 0.2577 \\ 8.062 \end{Bmatrix}$$

In a similar fashion, we find $T_{1,2}$, $T_{2,2}$, $T_{3,2}$ and $T_{1,3}$, $T_{2,3}$, $T_{3,3}$

$T_{1,2} = 6.168$, $T_{2,2} = 0.824$, $T_{3,2} = 4.236$

$T_{1,3} = 13.112$, $T_{2,3} = 8.321$, $T_{3,3} = 11.361$

this method can be repeated for any period of time.

End of Chapter 30!