Chapter 5: Series Solutions of ODE's

- previously, we have dealt with ODE's with constants and/or Cauchy-Euler ODE's. In chapter 5, we will mainly focus on ODE's with General variable coefficients.

- From calculus, recall the infinite power series:

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

where: (x) : is the series variable

$(a_0, a_1, \cdots, a_m)$ are the series coefficients "constants"

x_0 : Is the center of the series.

Generally, $x=0$, so:

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$$ "m: positive integer"

* 5.1 Solving ODE's Using Power Series method:

For a given ODE : $y'' + p(x)y' + q(x)y = 0$ "2nd order, linear, homog."

To Solve, let $\Rightarrow$ $y(x) = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$

So that

$$\begin{cases} y'(x) = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \\ y''(x) = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots \end{cases}$$

Note that m starts from 1 and 2

Then, Subst y, y', y'' in the ODE to obtain the Series coefficients and hence, the ODE solution

Example: Use power series to solve $y' = 2xy$ "1st order"

Sol. $y(x) = \sum_{m=0}^{\infty} a_m x^m$, and $y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$, subst.

$$\Rightarrow \sum_{m=1}^{\infty} m a_m x^{m-1} = 2x \sum_{m=0}^{\infty} a_m x^m \quad , \text{ expand!}$$

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots = 2x\left(a_0 + a_1 x + a_2 x^2 + \cdots\right)$$

Equate and collect terms $\{x^0, x, x^2, \ldots\}$, thus:-

constant (x^0): $a_1 = 0$

x: $2a_2 \cancel{x} = 2a_0 \cancel{x}$, $a_2 = a_0$

x^2: $3a_3 \cancel{x^2} = 2a_1 \cancel{x^2}$, $3a_3 = 2a_1 = 0$

x^3: $4a_4 \cancel{x^3} = 2a_2 \cancel{x^3}$, $4a_4 = 2a_2$, $a_4 = \frac{a_2}{2} = \frac{a_0}{2}$

So, the odd constants $a_1 = a_3 = a_5 \cdots = 0$ "Zero"

the even constants, $a_2 = a_0$, $a_4 = \frac{a_0}{2}$, $a_6 = \frac{a_4}{3} = \frac{a_0}{3 \times 2} = \frac{a_0}{3!}$

and so on, ...

therefore $y(x) = a_0\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right)$

a_0 = is arbitrary value "any real value"

Example: use power series method to solve: $y'' + y = 0$

Sol. $y = \sum_{m=0}^{\infty} a_m x^m$, $y'' = \sum_{m=2}^{\infty} m(m-1) x^{m-2}$, Subst. in ODE

$$\Rightarrow \underbrace{\sum_{m=2}^{\infty} a_m m(m-1) x^{m-2}}_{\text{let } m-2=s} + \underbrace{\sum_{m=0}^{\infty} a_m x^m}_{\text{let } m=s} = 0 \quad , \text{ so} \Rightarrow \sum_{s=0}^{\infty} a_s x^s = -\sum_{s=0} (s+1)(s+2) a_{s+2} x^s$$

$m = 2 \Rightarrow s = 0$

Therefore $a_{s+2} = -\frac{a_s}{(s+1)(s+2)} \rightarrow$

$s=0 \Rightarrow a_2 = \frac{-a_0}{(2)(1)}$

$s=1 \Rightarrow a_3 = \frac{-a_1}{(3)(2)}$

$s=2 \Rightarrow a_4 = \frac{-a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)(1)} = \frac{a_0}{4!}$

$s=3 \Rightarrow a_5 = \frac{-a_3}{(5)(4)} = \frac{a_1}{(5)(4)(3)(2)(1)!} = \frac{a_1}{5!}$

Therefore: $y(x) = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 - \frac{a_1}{5!} x^5 + \cdots$

OR:- $y(x) = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + a_1\left(x - \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots\right)$

↳ ODE Solution! { Practice Problem set 5-1 - page 174 ⇒ 8, 13, 14

5.2 Legendre's Equation and Legendre's polynomials

Legendre's Eq'n is a 2nd order linear ODE written as:-

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad , \quad n: \text{constant}$$

* To Solve this eq'n, we use $y = \sum_{m=0}^{\infty} a_m x^m$ { derive y', y'' Subst. in ODE and Let $k = n(n+1)$

$$\Rightarrow (1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0 \quad , \text{Eq}(1)$$

Term expansion & Distribution

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} a_m m(m-1)x^m - 2\sum_{m=1}^{\infty} m a_m x^m + k\sum_{m=0}^{\infty} a_m x^m = 0 \quad , \text{Eq}(2)$$

By expanding series terms:

$\boxed{m-2=s}$

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = (2)(1)a_2 + (3)(2)a_3 x + (4)(3)a_4 x^2 + \cdots + (s+2)(s+1)a_{s+2}x^s$$

$\boxed{s=m}$

$$\sum_{m=2}^{\infty} m(m-1)a_m x^m = (2)(1)a_2 x^2 + (3)(2)a_3 x^3 + (4)(3)a_4 x^4 + \cdots + (s)(s-1)a_s x^s$$

$\boxed{s=m}$

$$\sum_{m=1}^{\infty} m a_m x^m = (1)a_1 x + (2)a_2 x^2 + (3)a_3 x^3 + \cdots + (s)a_s x^s$$

$\boxed{s=m}$

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots + a_s x^s$$

substitute all in ODE of Eq(2)

$$(2)(1)a_2 + (3)(2)a_3 x + \cdots + (s+2)(s+1)a_{s+2}x^s - (2)(1)a_2 x^2 - (3)(2)a_3 x^3 - \cdots - (s)(s-1)a_s x^s$$

$$-2(1)a_1 x - 2(2)a_2 x^2 + \cdots + 2(s)a_s x^s + Ka_0 + Ka_1 x + \cdots + Ka_s x^s = 0$$

Now, collect terms and equate to "zero"

$$x^0 \text{ (constants)} \Rightarrow 2a_2 + \underbrace{(n)(n+1)}_{k}a_0 = 0 \quad \Rightarrow a_2 = \frac{-n(n+1)}{2}a_0$$

$$x^1 \Rightarrow 6a_3 - 2a_1 + n(n+1)a_1 = 0 \quad \Rightarrow a_3 = \frac{n(n+1)-2}{6}a_1$$

$$x^s \Rightarrow (s+2)(s+1)a_{s+2} - s(s-1)a_s + sa_s + \overbrace{n(n+1)}^{k}a_s = 0$$

So,

$$a_{s+2} = \frac{-(n-s)(n+s+1)}{(s+1)(s+2)}a_s \quad , s = 0, 1, 2, \ldots$$

Recursion formula

Eq(4) page 176 textbook

Tip: See example 3 page 169 For $n=1$

Therefore:

$s=0$

$$a_2 = \frac{-n(n+1)}{2!} a_0$$

$s=2$

$$a_4 = \frac{2+2+n(n+1)}{(4)(3)} a_2$$

$$a_4 = \frac{+(n-2)(n+1)(n+3)}{4!} a_0$$

$\vdots$

$s=1$

$$a_3 = \frac{-n(n+1)-1}{3!} a_1$$

$s=3$

$$a_5 = \frac{-(3)(2)-3+n(n+1)}{(5)(4)} a_3$$

$$a_5 = \frac{+(n-3)(n-1)(n+2)(n+4)}{5!} a_1$$

$\vdots$

Now, we will have two solutions, one for odd terms and one for even terms $\left(y = \sum_{m=0}^{\infty} a_m x^m\right)$, $m \to$ odd فردي, $\to$ Even زوجي

Even terms

$$y_1(x) = \sum_{\substack{m=\text{even} \\ (0,2,4,\ldots)}}^{\infty} a_m x^m = a_0 - a_0 \frac{(n+1)n}{2!} x^2 + a_0 \frac{n(n-2)(n+1)(n+3)}{4!} x^4 + \cdots$$

Odd terms

$$y_2(x) = \sum_{\substack{m=\text{odd} \\ 1,3,5,\ldots}}^{\infty} a_m x^m = a_1 x - a_1 \frac{(n-1)(n+2)}{3!} x^3 + a_0 \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 + \cdots$$

The ode final solution

$$y(x) = y_1(x) + y_2(x)$$

principle of superposition!

practice: Problem Set 5.2 - Page 179 $\Rightarrow$ "2", have a look on 13, 14, 15