Thus, the solution of the Fokker–Planck equation is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(-nt - \frac{1}{2}x^2\right) H_n\left(\frac{x}{\sqrt{2}}\right), \tag{7.8.29}$$

where H_n is the Hermite function and a_n are arbitrary constants to be determined from the given initial condition

$$u(x,0) = f(x). \tag{7.8.30}$$

We make the change of variables

$$\xi = x\,e^t \quad \text{and} \quad u = e^t v, \tag{7.8.31}$$

in equation (7.8.25). Consequently, equation (7.8.25) becomes

$$\frac{\partial v}{\partial t} = e^{2t}\,\frac{\partial^2 v}{\partial \xi^2}. \tag{7.8.32}$$

Making another change of variable t to $\tau(t)$, we transform (7.8.32) into the linear diffusion equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \xi^2}. \tag{7.8.33}$$

Finally, we note that the asymptotic behavior of the solution $u(x,t)$ as $t \to \infty$ is of special interest. The reader is referred to Reif (1965) for such behavior.

7.9 Exercises

1. Solve the following initial boundary-value problems:

(a)
$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < 1, \qquad t > 0,$$
$$u(x,0) = x(1-x), \qquad u_t(x,0) = 0, \qquad 0 \le x \le 1,$$
$$u(0,t) = u(1,t) = 0, \qquad t > 0.$$

(b)
$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < \pi, \qquad t > 0,$$
$$u(x,0) = 3\sin x, \qquad u_t(x,0) = 0, \qquad 0 \le x \le \pi,$$
$$u(0,t) = u(1,t) = 0, \qquad t > 0.$$

2. Determine the solutions of the following initial boundary-value problems:

(a)
$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < \pi, \qquad t > 0,$$
$$u(x,0) = 0, \qquad u_t(x,0) = 8\sin^2 x, \qquad 0 \le x \le \pi,$$
$$u(0,t) = u(\pi,t) = 0, \qquad t > 0.$$

(b)
$$u_{tt} = c^2 u_{xx} = 0, \qquad 0 < x < 1, \qquad t > 0,$$
$$u(x,0) = 0, \qquad u_t(x,0) = x\sin\pi x, \qquad 0 \le x \le 1,$$
$$u(0,t) = u(1,t) = 0, \qquad t > 0.$$

3. Find the solution of each of the following problems:

(a)
$$u_{tt} = c^2 u_{xx} = 0, \qquad 0 < x < 1, \qquad t > 0,$$
$$u(x,0) = x(1-x), \qquad u_t(x,0) = x - \tan\frac{\pi x}{4}, \qquad 0 \le x \le 1,$$
$$u(0,t) = u(\pi,t) = 0, \qquad t > 0.$$

(b)
$$u_{tt} = c^2 u_{xx} = 0, \qquad 0 < x < \pi, \qquad t > 0,$$
$$u(x,0) = \sin x, \qquad u_t(x,0) = x^2 - \pi x, \qquad 0 \le x \le \pi,$$
$$u(0,t) = u(\pi,t) = 0, \qquad t > 0.$$

4. Solve the following problems:

(a)
$$u_{tt} = c^2 u_{xx} = 0, \qquad 0 < x < \pi, \qquad t > 0,$$
$$u(x,0) = x + \sin x, \qquad u_t(x,0) = 0, \qquad 0 \le x \le \pi,$$
$$u(0,t) = u_x(\pi,t) = 0, \qquad t > 0.$$

(b)
$$u_{tt} = c^2 u_{xx} = 0, \qquad 0 < x < \pi, \qquad t > 0,$$
$$u(x,0) = \cos x, \qquad u_t(x,0) = 0, \qquad 0 \le x \le \pi,$$
$$u_x(0,t) = 0, \qquad u_x(\pi,t) = 0, \qquad t > 0.$$

5. By the method of separation of variables, solve the telegraph equation:

$$\begin{aligned} u_{tt} + a u_t + b u &= c^2 u_{xx}, \qquad 0 < x < l, \qquad t > 0, \\ u(x,0) &= f(x), \qquad u_t(x,0) = 0, \\ u(0,t) &= u(l,t) = 0, \qquad t > 0. \end{aligned}$$

6. Obtain the solution of the damped wave motion problem:

$$\begin{aligned} u_{tt} + a u_t &= c^2 u_{xx}, \qquad 0 < x < l, \qquad t > 0, \\ u(x,0) &= 0, \qquad u_t(x,0) = g(x), \\ u(0,t) &= u(l,t) = 0. \end{aligned}$$

7. The torsional oscillation of a shaft of circular cross section is governed by the partial differential equation

$$\theta_{tt} = a^2 \theta_{xx},$$

where $\theta(x,t)$ is the angular displacement of the cross section and a is a physical constant. The ends of the shaft are fixed elastically, that is,

$$\theta_x(0,t) - h\,\theta(0,t) = 0, \qquad \theta_x(l,t) + h\,\theta(l,t) = 0.$$

Determine the angular displacement if the initial angular displacement is $f(x)$.

8. Solve the initial boundary-value problem of the longitudinal vibration of a truncated cone of length l and base of radius a. The equation of motion is given by

$$\left(1 - \frac{x}{h}\right)^2 \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial}{\partial x}\left[\left(1 - \frac{x}{h}\right)^2 \frac{\partial u}{\partial x}\right], \qquad 0 < x < l, \quad t > 0,$$

where $c^2 = (E/\rho)$, E is the elastic modulus, ρ is the density of the material and $h = la/(a - l)$. The two ends are rigidly fixed. If the initial displacement is $f(x)$, that is, $u(x,0) = f(x)$, find $u(x,t)$.

9. Establish the validity of the formal solution of the initial boundary-value problems:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad 0 < x < \pi, \qquad t > 0, \\ u(x,0) &= f(x), \qquad u_t(x,0) = g(x), \qquad 0 \le x \le \pi, \\ u_x(0,t) &= 0, \qquad u_x(\pi,t) = 0, \qquad t > 0. \end{aligned}$$

10. Prove the uniqueness of the solution of the initial boundary-value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad 0 < x < \pi, \qquad t > 0, \\ u(x,0) &= f(x), \qquad u_t(x,0) = g(x), \qquad 0 \le x \le \pi, \\ u_x(0,t) &= 0, \qquad u_x(\pi,t) = 0, \qquad t > 0. \end{aligned}$$

11. Determine the solution of

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + A \sinh x, \quad 0 < x < l, \quad t > 0, \\ u(x,0) &= 0, \quad u_t(x,0) = 0, \quad 0 \le x \le l, \\ u(0,t) &= h, \quad u(l,t) = k, \quad t > 0, \end{aligned}$$

where h, k, and A are constants.

12. Solve the problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + Ax, \quad 0 < x < 1, \quad t > 0, \quad A = \text{constant}, \\ u(x,0) &= 0, \quad u_t(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) &= 0, \quad u(1,t) = 0, \quad t > 0. \end{aligned}$$

13. Solve the problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + x^2, \quad 0 < x < 1, \quad t > 0, \\ u(x,0) &= x, \quad u_t(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) &= 0, \quad u(1,t) = 1, \quad t \ge 0. \end{aligned}$$

14. Find the solution of the following problems:

(a)
$$\begin{aligned} u_t &= k u_{xx} + h, \quad 0 < x < 1, \quad t > 0, \quad h = \text{constant}, \\ u(x,0) &= u_0 (1 - \cos \pi x), \quad 0 \le x \le 1, \quad u_0 = \text{constant}, \\ u(0,t) &= 0, \quad u(l,t) = 2u_0, \quad t \ge 0. \end{aligned}$$

(b)
$$\begin{aligned} u_t &= k u_{xx} - hu, \quad 0 < x < l, \quad t > 0, \quad h = \text{constant}, \\ u(x,0) &= f(x), \quad 0 \le x \le l, \\ u_x(0,t) &= u_x(l,t) = 0, \quad t > 0. \end{aligned}$$

15. Obtain the solution of each of the following initial boundary-value problems:

(a)
$$\begin{aligned} u_t &= 4\, u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u(x,0) &= x^2 (1 - x), \quad 0 \le x \le 1, \\ u(0,t) &= 0, \quad u(l,t) = 0, \quad t \ge 0. \end{aligned}$$

(b)
$$\begin{aligned} u_t &= k\, u_{xx}, \quad 0 < x < \pi, \quad t > 0, \\ u(x,0) &= \sin^2 x, \quad 0 \le x \le \pi, \\ u(0,t) &= 0, \quad u(\pi,t) = 0, \quad t \ge 0. \end{aligned}$$

(c)
$$\begin{aligned} u_t &= u_{xx}, && 0 < x < 2, && t > 0, \\ u(x,0) &= x, && 0 \le x \le 2, && \\ u(0,t) &= 0, && u_x(2,t) = 1, && t \ge 0. \end{aligned}$$

(d)
$$\begin{aligned} u_t &= k\,u_{xx}, && 0 < x < l, && t > 0, \\ u(x,0) &= \sin(\pi x/2l), && 0 \le x \le l, && \\ u(0,t) &= 0, && u(l,t) = 1, && t \ge 0. \end{aligned}$$

16. Find the temperature distribution in a rod of length l. The faces are insulated, and the initial temperature distribution is given by $x(l-x)$.

17. Find the temperature distribution in a rod of length π, one end of which is kept at zero temperature and the other end of which loses heat at a rate proportional to the temperature at that end $x = \pi$. The initial temperature distribution is given by $f(x) = x$.

18. The voltage distribution in an electric transmission line is given by
$$v_t = k\,v_{xx}, \quad 0 < x < l, \quad t > 0.$$
A voltage equal to zero is maintained at $x = l$, while at the end $x = 0$, the voltage varies according to the law
$$v(0,t) = Ct, \qquad t > 0,$$
where C is a constant. Find $v(x,t)$ if the initial voltage distribution is zero.

19. Establish the validity of the formal solution of the initial boundary-value problem:
$$\begin{aligned} u_t &= k\,u_{xx}, && 0 < x < l, && t > 0, \\ u(x,0) &= f(x), && 0 \le x \le l, && \\ u(0,t) &= 0, && u_x(l,t) = 0, && t \ge 0. \end{aligned}$$

20. Prove the uniqueness of the solution of the problem:
$$\begin{aligned} u_t &= k\,u_{xx}, && 0 < x < l, && t > 0, \\ u(x,0) &= f(x), && 0 \le x \le l, && \\ u_x(0,t) &= 0, && u_x(l,t) = 0, && t \ge 0. \end{aligned}$$

21. Solve the radioactive decay problem:

$$\begin{aligned} u_t - k\,u_{xx} &= Ae^{-ax}, && 0 < x < \pi, && t > 0, \\ u(x,0) &= \sin x, && 0 \le x \le \pi, \\ u(0,t) &= 0, && u(\pi,t) = 0, && t \ge 0. \end{aligned}$$

22. Determine the solution of the initial boundary-value problem:

$$\begin{aligned} u_t - k\,u_{xx} &= h(x,t), && 0 < x < l, && t > 0, \quad k = \text{constant}, \\ u(x,0) &= f(x), && 0 \le x \le l, \\ u(0,t) &= p(t), && u(l,t) = q(t), && t \ge 0. \end{aligned}$$

23. Determine the solution of the initial boundary-value problem:

$$\begin{aligned} u_t - k\,u_{xx} &= h(x,t), && 0 < x < l, && t > 0, \\ u(x,0) &= f(x), && 0 \le x \le l, \\ u(0,t) &= p(t), && u_x(l,t) = q(t), && t \ge 0. \end{aligned}$$

24. Solve the problem:

$$\begin{aligned} u_t - k\,u_{xx} &= 0, && 0 < x < 1, && t > 0, \\ u(x,0) &= x(1-x), && 0 \le x \le 1, \\ u(0,t) &= t, && u(1,t) = \sin t, && t \ge 0. \end{aligned}$$

25. Solve the problem:

$$\begin{aligned} u_t - 4u_{xx} &= xt, && 0 < x < 1, && t \ge 0, \\ u(x,0) &= \sin \pi x, && 0 \le x \le 1, \\ u(0,t) &= t, && u(1,t) = t^2, && t \ge 0. \end{aligned}$$

26. Solve the problem:

$$\begin{aligned} u_t - k\,u_{xx} &= x \cos t, && 0 < x < \pi, && t > 0, \\ u(x,0) &= \sin x, && 0 \le x \le \pi, \\ u(0,t) &= t^2, && u(\pi,t) = 2t, && t \ge 0. \end{aligned}$$

27. Solve the problem:

$$\begin{aligned} u_t - u_{xx} &= 2x^2 t, && 0 < x < 1, && t > 0, \\ u(x,0) &= \cos(3\pi x/2), && 0 \le x \le 1, \\ u(0,t) &= 1, && u_x(1,t) = \frac{3\pi}{2}, && t \ge 0. \end{aligned}$$

28. Solve the problem:

$$\begin{aligned} u_t - 2\,u_{xx} &= h, && 0 < x < 1, && t > 0, \quad h = \text{constant}, \\ u(x,0) &= x, && 0 \le x \le 1, \\ u(0,t) &= \sin t, && u_x(1,t) + u(1,t) = 2, && t \ge 0. \end{aligned}$$

29. Determine the solution of the initial boundary-value problem:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= h(x,t), & 0 < x < l, & \quad t > 0, \\ u(x,0) &= f(x), & 0 \le x \le l, & \\ u_t(x,0) &= g(x), & 0 \le x \le l, & \\ u(0,t) &= p(t), & u_x(l,t) = q(t), & \quad t \ge 0. \end{aligned}$$

30. Determine the solution of the initial boundary-value problem:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= h(x,t), & 0 < x < l, & \quad t > 0, \\ u(x,0) &= f(x), & 0 \le x \le l, & \\ u_t(x,0) &= g(x), & 0 \le x \le l, & \\ u_x(0,t) &= p(t), & u_x(l,t) = q(t), & \quad t \ge 0. \end{aligned}$$

31. Solve the problem:

$$\begin{aligned} u_{tt} - u_{xx} &= 0, & 0 < x < 1, & \quad t > 0, \\ u(x,0) &= x, & u_t(x,0) = 0, & \quad 0 \le x \le 1, \\ u(0,t) &= t^2, & u(1,t) = \cos t, & \quad t \ge 0. \end{aligned}$$

32. Solve the problem:

$$\begin{aligned} u_{tt} - 4\,u_{xx} &= xt, & 0 < x < 1, & \quad t > 0, \\ u(x,0) &= x, & u_t(x,0) = 0, & \quad 0 \le x \le 1, \\ u(0,t) &= 0, & u_x(1,t) = 1 + t, & \quad t \ge 0. \end{aligned}$$

33. Solve the problem:

$$\begin{aligned} u_{tt} - 9\,u_{xx} &= 0, & 0 < x < 1, & \quad t > 0, \\ u(x,0) &= \sin\left(\frac{\pi x}{2}\right), & u_t(x,0) = 1 + x, & \quad 0 \le x \le 1, \\ u_x(0,t) &= \pi/2, & u_x(1,t) = 0, & \quad t \ge 0. \end{aligned}$$

34. Find the solution of the problem:

$$\begin{aligned} u_{tt} + 2k\,u_t - c^2 u_{xx} &= 0, & 0 < x < l, & \quad t > 0, \\ u(x,0) &= 0, & u_t(x,0) = 0, & \quad 0 \le x \le l, \\ u_x(0,t) &= 0, & u(l,t) = h, & \quad t \ge 0, \quad h = \text{constant}. \end{aligned}$$

35. Solve the problem:

$$\begin{aligned} u_t - c^2 u_{xx} + hu &= hu_0, & -\pi < x < \pi, & \quad t > 0, \\ u(x,0) &= f(x), & -\pi \le x \le \pi, & \\ u(-\pi,t) &= u(\pi,t), & u_x(-\pi,t) = u_x(\pi,t), & \quad t \ge 0, \end{aligned}$$

where h and u_0 are constants.