where C_3 is the combined path: from A_1 to A_2 via C_1 and from A_2 to A_3 via C_2;

$$\int_{A \atop C}^{B} P\,dx + Q\,dy = -\int_{B \atop C'}^{A} P\,dx + Q\,dy, \tag{5.19}$$

where C' denotes C traced in the reverse direction;

$$\int\limits_C (P_1\,dx + Q_1\,dy) + \int\limits_C (P_2\,dx + Q_2\,dy) = \int\limits_C (P_1 + P_2)\,dx + (Q_1 + Q_2)\,dy; \tag{5.20}$$

$$K \int\limits_C (P\,dx + Q\,dy) = \int\limits_C (KP)\,dx + (KQ)\,dy, \quad K = \text{const}; \tag{5.21}$$

$$\int\limits_C ds = L = \text{length of } C; \tag{5.22}$$

if $|f(x, y)| \leq M$ on C, then

$$\left| \int\limits_C f(x, y)\,ds \right| \leq M \cdot L; \tag{5.23}$$

if C is a simple closed curve, as in Fig. 5.6(a) then

$$\oint\limits_C x\,dy = -\oint\limits_C y\,dx = \text{area enclosed by } C. \tag{5.24}$$

These theorems all hold under the assumptions that the paths are piecewise smooth and that the functions being integrated are continuous. All except the formula (5.24) follow at once from appropriate parametric representations; (5.24) will be proved in Section 5.5.

PROBLEMS

1. Evaluate the following integrals along the straight-line paths joining the end points:

a) $\int_{(0,0)}^{(2,2)} y^2\,dx$,

b) $\int_{(2,1)}^{(1,2)} y\,dx$,

c) $\int_{(1,1)}^{(2,1)} x\,dy$.

2. Evaluate the following line integrals:

a) $\int_{(0,-1)\,C}^{(0,1)} y^2\,dx + x^2\,dy$, where C is the semicircle $x = \sqrt{1 - y^2}$;

b) $\int_{(0,0)\,C}^{(2,4)} y\,dx + x\,dy$, where C is the parabola $y = x^2$;

c) $\int_{(1,0)\,C}^{(0,1)} \frac{y\,dx - x\,dy}{x^2 + y^2}$, where C is the curve $x = \cos^3 t,\ y = \sin^3 t,\ 0 \leq t \leq \frac{\pi}{2}$.

(Hint: Set $u = \tan^3 t$ in the integral for t.)

3. Evaluate the following line integrals:

a) $\oint_C y^2\,dx + xy\,dy$, where C is the square with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$;

b) $\oint_C y\,dx - x\,dy$, where C is the circle $x^2 + y^2 = 1$ (cf. (5.24));

c) $\oint_C x^2y^2\,dx - xy^3\,dy$, where C is the triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$.

4. Evaluate the following line integrals:

a) $\oint_C (x^2 - y^2)\,ds$, where C is the circle $x^2 + y^2 = 4$;

b) $\int_{(0,0)\,C}^{(1,1)} x\,ds$, where C is the line $y = x$;

c) $\int_{(0,0)\,C}^{(1,1)} ds$, where C is the parabola $y = x^2$.

5. Let a path (5.1) be given and let a *change of parameter* be made by an equation $t = g(\tau)$, $\alpha \le \tau \le \beta$, where $g'(\tau)$ is continuous and positive in the interval and $g(\alpha) = h$, $g(\beta) = k$. As in (5.4) the line integral $\int f(x, y)\,dx$ on the path $x = \phi(g(\tau))$, $y = \psi(g(\tau))$ is given by

$$\int_\alpha^\beta f[\phi(g(\tau)), \psi(g(\tau))]\frac{d}{d\tau}\phi(g(\tau))\,d\tau.$$

Show that this equals the integral in (5.4), so that such a change of parameter does not affect the value of the line integral.

6. (*Numerical evaluation of line integrals*) If the parametric equations of the path C are explicitly known, Eqs. (5.4) and (5.5) reduce the evaluation of line integrals to a problem in ordinary definite integrals, to which the methods of Section 4.1 apply. One can in any case evaluate the integral directly from its definition as limit of a sum. Thus

$$\int_C P\,dx + Q\,dy \sim \sum_{i=1}^{n}(P(x_i^*, y_i^*)\,\Delta_i x + Q(x_i^*, y_i^*)\,\Delta_i y).$$

The points (x_i^*, y_i^*) can be chosen as the subdivision points (x_{i-1}, y_{i-1}) or as (x_i, y_i); one can also use a trapezoidal rule to obtain the sum:

$$\sum_{i=1}^{n}\left\{\frac{1}{2}[P(x_{i-1}, y_{i-1}) + P(x_i, y_i)]\Delta_i x + \frac{1}{2}[Q(x_{i-1}, y_{i-1}) + Q(x_i, y_i)]\Delta_i y\right\}. \qquad \text{(a)}$$

For example,

$$\int_{(0,0)\,C}^{(2,2)} y^2\,dx + x^2\,dy \sim \left[\frac{1}{2}(0+1)\cdot 1 + \frac{1}{2}(0+1)\cdot 1\right] + \left[\frac{1}{2}(1+4)\cdot 1 + \frac{1}{2}(1+4)\cdot 1\right] = 6,$$

where C is the straight line segment from $(0, 0)$ to $(2, 2)$ and the subdivision points are $(0, 0)$, $(1, 1)$, $(2, 2)$.

Let the functions P and Q be given by the following table at the points $A \ldots S$.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	S
x	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
y	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
P	0	3	8	5	3	0	5	2	8	5	0	1	2	7	3	4
Q	1	2	3	4	2	4	6	8	3	6	9	2	4	8	2	6