15

Tables of Integral Transforms

In this chapter we provide a set of *short* tables of integral transforms of the functions that are either cited in the text or are in most common use in mathematical, physical, and engineering applications. For exhaustive lists of integral transforms, the reader is referred to Erdélyi et al. (1954), Campbell and Foster (1948), Ditkin and Prudnikov (1965), Doetsch (1970), Marichev (1983), Debnath (1995), and Oberhettinger (1972).

15.1 Fourier Transforms

	$f(x)$	$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-ikx) f(x)\, dx$
1	$\exp(-a\lvert x\rvert), \qquad a > 0$	$\left(\sqrt{\frac{2}{\pi}}\right) a \left(a^2 + k^2\right)^{-1}$
2	$x \exp(-a\lvert x\rvert), \qquad a > 0$	$\left(\sqrt{\frac{2}{\pi}}\right) (-2aik) \left(a^2 + k^2\right)^{-2}$
3	$\exp\left(-ax^2\right), \qquad a > 0$	$\frac{1}{\sqrt{2a}} \exp\left(-\frac{k^2}{4a}\right)$
4	$\left(x^2 + a^2\right)^{-1}, \qquad a > 0$	$\sqrt{\frac{\pi}{2}}\ \frac{\exp(-a\lvert k\rvert)}{a}$
5	$x\left(x^2 + a^2\right)^{-1}$	$\sqrt{\frac{\pi}{2}} \left(\frac{ik}{2a}\right) \exp(-a\lvert k\rvert)$

	$f(x)$	$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-ikx)\,f(x)\,dx$
6	$\begin{cases} c, & a\le x\le b \\ 0, & \text{outside.} \end{cases}$	$\frac{ic}{\sqrt{2\pi}}\frac{1}{k}\left(e^{-ibk}-e^{-iak}\right)$
7	$\lvert x\rvert\exp(-a\lvert x\rvert), \qquad a>0$	$\sqrt{\frac{2}{\pi}}\left(a^2-k^2\right)\left(a^2+k^2\right)^{-2}$
8	$\frac{\sin ax}{x}$	$\sqrt{\frac{\pi}{2}}\,H(a-\lvert k\rvert)$
9	$\exp\{-x(a-i\omega)\}\,H(x)$	$\frac{1}{\sqrt{2\pi}}\frac{i}{(\omega-k+ia)}$
10	$\left(a^2-x^2\right)^{-\frac{1}{2}}\,H(a-\lvert x\rvert)$	$\sqrt{\frac{\pi}{2}}\,J_0(ak)$
11	$\frac{\sin\left[b\left(x^2+a^2\right)^{\frac{1}{2}}\right]}{\left(x^2+a^2\right)^{\frac{1}{2}}}$	$\sqrt{\frac{\pi}{2}}\,J_0\left(a\sqrt{b^2-k^2}\right)H(b-\lvert k\rvert)$
12	$\frac{\cos\left(b\sqrt{a^2-x^2}\right)}{\left(a^2-x^2\right)^{\frac{1}{2}}}\,H(a-\lvert x\rvert)$	$\sqrt{\frac{\pi}{2}}\,J_0\left(a\sqrt{b^2+k^2}\right)$
13	$e^{-ax}\,H(x), \qquad a>0$	$\frac{1}{\sqrt{2\pi}}(a-ik)\left(a^2+k^2\right)^{-1}$
14	$\frac{1}{\sqrt{\lvert x\rvert}}\exp(-a\lvert x\rvert), \qquad a>0$	$\left(a^2+k^2\right)^{-\frac{1}{2}}\left[a+\left(a^2+k^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$
15	$\delta^{(n)}(x-a),\ n=0,1,2,\ldots$	$\frac{1}{\sqrt{2\pi}}(ik)^n\exp(-iak)$
16	$\exp(iax)$	$\sqrt{2\pi}\,\delta(k-a)$

15.2 Fourier Sine Transforms

	$f(x)$	$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(kx) f(x)\, dx$
1	$\exp(-ax), \quad a > 0$	$\sqrt{\frac{2}{\pi}} k \left(a^2 + k^2\right)^{-1}$
2	$x \exp(-ax), \quad a > 0$	$\sqrt{\frac{2}{\pi}} (2ak) \left(a^2 + k^2\right)^{-2}$
3	$x^{\alpha-1}, \quad 0 < \alpha < 1$	$\sqrt{\frac{2}{\pi}}\ k^{-\alpha} \Gamma(\alpha) \sin\left(\frac{\pi\alpha}{2}\right)$
4	$\frac{1}{\sqrt{x}}$	$\frac{1}{\sqrt{k}}, \quad k > 0$
5	$x^{\alpha-1} e^{-ax}, \quad \alpha > -1,$ $a > 0$	$\sqrt{\frac{2}{\pi}}\ \Gamma(\alpha)\, r^{-\alpha} \sin(\alpha\theta),$ where $r = \left(a^2 + k^2\right)^{\frac{1}{2}}, \quad \theta = \tan^{-1}\left(\frac{k}{a}\right)$
6	$x^{-1} e^{-ax}, \quad a > 0$	$\sqrt{\frac{2}{\pi}}\ \tan^{-1}\left(\frac{k}{a}\right), \quad k > 0$
7	$x \exp\left(-a^2 x^2\right)$	$2^{-3/2} \left(\frac{k}{a^3}\right) \exp\left(-\frac{k^2}{4a^2}\right)$
8	$\operatorname{erfc}(ax)$	$\sqrt{\frac{2}{\pi}}\ \frac{1}{k} \left[1 - \exp\left(-\frac{k^2}{4a^2}\right)\right]$
9	$x \left(a^2 + x^2\right)^{-1}$	$\sqrt{\frac{\pi}{2}} \exp(-ak), \quad a > 0$
10	$x \left(a^2 + x^2\right)^{-2}$	$\frac{1}{\sqrt{2\pi}} \left(\frac{k}{a}\right) \exp(-ak), \quad (a > 0)$

	$f(x)$	$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(kx) f(x)\, dx$
11	$H(a-x), \quad a>0$	$\sqrt{\frac{2}{\pi}}\ \frac{1}{k}(1-\cos ak)$
12	$x^{-1}J_0(ax)$	$\begin{cases} \sqrt{\frac{2}{\pi}}\ \sin^{-1}\left(\frac{k}{a}\right), & 0<k<a \\ \sqrt{\frac{\pi}{2}}, & a<k<\infty \end{cases}$
13	$x\left(a^2+x^2\right)^{-1} J_0(bx),$ $a>0,\ b>0$	$\sqrt{\frac{\pi}{2}}\ e^{-ak} I_0(ab), \quad a<k<\infty$
14	$J_0(a\sqrt{x}), \quad a>0$	$\sqrt{\frac{2}{\pi}}\ \frac{1}{k}\cos\left(\frac{a^2}{4k}\right)$
15	$\left(x^2-a^2\right)^{\nu-\frac{1}{2}} H(x-a),$ $\lvert\nu\rvert<\frac{1}{2}$	$2^{\nu-\frac{1}{2}}\left(\frac{a}{k}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right) J_{-\nu}(ak)$
16	$x^{1-\nu}\left(x^2+a^2\right)^{-1} J_\nu(ax),$ $\nu>-\frac{3}{2}, \quad a,b>0$	$\sqrt{\frac{\pi}{2}}\ a^{-\nu}\exp(-ak)\, I_\nu(ab),$ $a<k<\infty$
17	$x^{-\nu} J_{\nu+1}(ax), \quad \nu>-\frac{1}{2}$	$\frac{k\left(a^2-k^2\right)^{\nu-\frac{1}{2}}}{2^{\nu-\frac{1}{2}} a^{\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)} H(a-k)$
18	$\operatorname{erfc}(ax)$	$\sqrt{\frac{2}{\pi}}\ \frac{1}{k}\left[1-\exp\left(-\frac{k^2}{4a^2}\right)\right]$
19	$x^{-\alpha}, \quad 0<\operatorname{Re}\alpha<2$	$\sqrt{\frac{2}{\pi}}\,\Gamma(1-\alpha)\, k^{\alpha-1}\cos\left(\frac{\alpha\pi}{2}\right)$
20	$\left(ax-x^2\right)^{\alpha-\frac{1}{2}} H(a-x),$ $\alpha>-\frac{1}{2}$	$\sqrt{2}\,\Gamma\left(\alpha+\frac{1}{2}\right)\left(\frac{a}{k}\right)^\alpha \sin\left(\frac{ak}{2}\right) J_\alpha\left(\frac{ak}{2}\right)$

15.3 Fourier Cosine Transforms

	$f(x)$	$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(kx) f(x)\, dx$
1	$\exp(-ax), \quad a > 0$	$\left(\sqrt{\frac{2}{\pi}}\right) a \left(a^2 + k^2\right)^{-1}$
2	$x \exp(-ax), \qquad a > 0$	$\left(\sqrt{\frac{2}{\pi}}\right) \left(a^2 - k^2\right) \left(a^2 + k^2\right)^{-2}$
3	$\exp\left(-a^2 x^2\right)$	$\frac{1}{a\sqrt{2}} \exp\left(-\frac{k^2}{4a^2}\right)$
4	$H(a - x)$	$\sqrt{\frac{2}{\pi}} \left(\frac{\sin ak}{k}\right)$
5	$x^{a-1}, \quad 0 < a < 1$	$\sqrt{\frac{2}{\pi}}\, \Gamma(a)\, k^{-a} \cos\left(\frac{a\pi}{2}\right)$
6	$\cos\left(ax^2\right), \quad a > 0$	$\frac{1}{2\sqrt{a}} \left[\cos\left(\frac{k^2}{4a}\right) + \sin\left(\frac{k^2}{4a}\right)\right]$
7	$\sin\left(ax^2\right), \quad a > 0$	$\frac{1}{2\sqrt{a}} \left[\cos\left(\frac{k^2}{4a}\right) - \sin\left(\frac{k^2}{4a}\right)\right]$
8	$\left(a^2 - x^2\right)^{\nu - \frac{1}{2}}\ H(a - x),\ \nu > -\frac{1}{2}$	$2^{\nu - \frac{1}{2}}\, \Gamma\left(\nu + \frac{1}{2}\right) \left(\frac{a}{k}\right)^\nu J_\nu(ak)$
9	$\left(a^2 + x^2\right)^{-1}\ J_0(bx),\ a, b > 0$	$\sqrt{\frac{\pi}{2}}\, a^{-1} \exp(-ak)\, I_0(ab)$, $b < k < \infty$
10	$x^{-\nu} J_\nu(ax),\ \nu > -\frac{1}{2}$	$\frac{\left(a^2 - k^2\right)^{\nu - \frac{1}{2}} H(a-k)}{2^{\nu - \frac{1}{2}} a^\nu \Gamma\left(\nu + \frac{1}{2}\right)}$

	$f(x)$	$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(kx) f(x)\, dx$
11	$\left(x^2+a^2\right)^{-\frac{1}{2}} \exp\left[-b\left(x^2+a^2\right)^{\frac{1}{2}}\right]$	$K_0\left[a\left(k^2+b^2\right)^{\frac{1}{2}}\right],\ a>0,\ b>0$
12	$x^{\nu-1}e^{-ax}, \quad \nu>0,\ a>0$	$\sqrt{\frac{2}{\pi}}\, \Gamma(\nu)\, r^{-\nu} \cos n\theta$, where $r=\left(a^2+k^2\right)^{\frac{1}{2}}$, $\theta = \tan^{-1}\left(\frac{k}{a}\right)$
13	$\frac{2}{x}\, e^{-x} \sin x$	$\sqrt{\frac{2}{\pi}}\ \tan^{-1}\left(\frac{2}{k^2}\right)$
14	$\sin\left[a\left(b^2-x^2\right)^{\frac{1}{2}} H(b-x)\right]$	$\sqrt{\frac{\pi}{2}}\ (ab)\left(a^2+k^2\right)^{-\frac{1}{2}} \times J_1\left[b\left(a^2+k^2\right)^{\frac{1}{2}}\right]$
15	$\frac{\left(1-x^2\right)}{(1+x^2)^2}$	$\sqrt{\frac{\pi}{2}}\, k \exp(-k)$
16	$x^{-\alpha}, \quad 0<\alpha<1$	$\sqrt{\frac{\pi}{2}}\ \frac{k^{\alpha-1}}{\Gamma(\alpha)} \sec\left(\frac{\pi\alpha}{2}\right)$
17	$\left(\frac{1}{a}+x\right) e^{-ax}, \quad a>0$	$\sqrt{\frac{\pi}{2}}\ \frac{2a^2}{(a^2+k^2)^2}$
18	$\log\left(1+\frac{a^2}{x^2}\right), \quad a>0$	$\sqrt{2\pi}\ \frac{\left(1-e^{-ak}\right)}{k}$
19	$\log\left(\frac{a^2+x^2}{b^2+x^2}\right), \quad a,b>0$	$\sqrt{2\pi}\ \frac{\left(e^{-bk}-e^{-ak}\right)}{k}$
20	$a\left(x^2+a^2\right)^{-1}, \quad a>0$	$\sqrt{\frac{\pi}{2}}\ \exp(-ak), \quad k>0$

15.4 Laplace Transforms

	$f(t)$	$\overline{f}(s) = \int_0^\infty \exp(-st) f(t)\, dt$
1	$f^{(n)}(t)$	$s^n \overline{f}(s) - \sum_{r=0}^{n-1} s^{n-r-1} f^{(r)}(0)$
2	$\int_0^t f(t-\tau) g(\tau)\, d\tau$	$\overline{f}(s)\, \overline{g}(s)$
3	$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} \overline{f}(s)$
4	$f(t-a) H(t-a)$	$\exp(-as)\, \overline{f}(s)$
5	$t^n \qquad (n = 0, 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
6	e^{at}	$\frac{1}{s-a}$
7	$t^n e^{-at}$	$\frac{\Gamma(n+1)}{(s+a)^{n+1}}$
8	$t^a \qquad (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$
9	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
10	$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$
11	$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
12	$2\sqrt{t}$	$\frac{1}{s}\sqrt{\frac{\pi}{s}}$

	$f(t)$	$\overline{f}(s) = \int_0^\infty \exp(-st) f(t)\, dt$
13	$t^{-1/2} \exp\left(-\frac{a}{t}\right)$	$\sqrt{\frac{\pi}{s}} \exp\left(-2\sqrt{as}\right)$
14	$t^{-3/2} \exp\left(-\frac{a}{t}\right)$	$\sqrt{\frac{\pi}{a}} \exp\left(-2\sqrt{as}\right)$
15	$\frac{1}{\sqrt{\pi t}} (1 + 2at) e^{at}$	$\frac{s}{(s-a)\sqrt{s-a}}$
16	$\frac{1}{2\sqrt{\pi t^3}} \left(e^{bt} - e^{at}\right)$	$\sqrt{s-a} - \sqrt{s-b}$
17	$\exp\left(a^2 t\right) \operatorname{erf}\left(a\sqrt{t}\right)$	$\frac{a}{\sqrt{s}(s-a^2)}$
18	$\exp\left(a^2 t\right) \operatorname{erfc}\left(a\sqrt{t}\right)$	$\frac{1}{\sqrt{s}\left(\sqrt{s}+a\right)}$
19	$\frac{1}{\sqrt{\pi t}} + a \exp\left(a^2 t\right) \operatorname{erf}\left(a\sqrt{t}\right)$	$\frac{\sqrt{s}}{(s-a^2)}$
20	$\frac{1}{\sqrt{\pi t}} - a \exp\left(a^2 t\right) \operatorname{erfc}\left(a\sqrt{t}\right)$	$\frac{1}{\sqrt{s}+a}$
21	$\frac{\exp(-at)}{\sqrt{b}-a} \operatorname{erf}\left(\sqrt{(b-a)\, t}\right)$	$\frac{1}{(s+a)\sqrt{s+b}}$
22	$\frac{1}{2} e^{i\omega t} \left[\exp(-\lambda z) \operatorname{erfc}\left(\zeta - \sqrt{i\omega t}\right) + \exp(\lambda z) \operatorname{erfc}\left(\zeta + \sqrt{i\omega t}\right)\right]$, where $\zeta = z/2\sqrt{\nu t}, \quad \lambda = \sqrt{\frac{i\omega}{\nu}}$	$(s - i\omega)^{-1} \exp\left(-z\sqrt{\frac{s}{\nu}}\right)$
23	$\frac{1}{2} \left[\exp(-ab) \operatorname{erfc}\left(\frac{b-2at}{2\sqrt{t}}\right) + \exp(ab) \operatorname{erfc}\left(\frac{b+2at}{2\sqrt{t}}\right)\right]$	$\exp\left[-b\left(s + a^2\right)^{\frac{1}{2}}\right]$

	$f(t)$	$\overline{f}(s) = \int_0^\infty \exp(-st) f(t)\, dt$
24	$J_0(at)$	$\left(s^2 + a^2\right)^{-\frac{1}{2}}$
25	$I_0(at)$	$\left(s^2 - a^2\right)^{-\frac{1}{2}}$
26	$t^{\alpha-1} \exp(-at), \qquad \alpha > 0$	$\Gamma(\alpha)(s+a)^{-\alpha}$
27	$t^{-1} J_\nu(at)$	$\nu^{-1} a^\nu \left(\sqrt{s^2+a^2} + s\right)^{-\nu}$, $\operatorname{Re}\nu > -\frac{1}{2}$
28	$J_0\left(a\sqrt{t}\right)$	$\frac{1}{s} \exp\left(-\frac{a^2}{4s}\right)$
29	$\left(\frac{2}{a}\right)^\nu t^{\nu/2} J_\nu\left(a\sqrt{t}\right)$	$s^{-(\nu+1)} \exp\left(-\frac{a^2}{4s}\right)$, $\operatorname{Re}\nu > -\frac{1}{2}$
30	$\frac{a}{2t\sqrt{\pi t}} \exp\left(-\frac{a^2}{4t}\right)$	$\exp(-a\sqrt{s}), \quad a > 0$
31	$\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{a^2}{4t}\right)$	$\frac{1}{\sqrt{s}} \exp(-a\sqrt{s}), \quad a \geq 0$
32	$\exp\left(-\frac{a^2 t^2}{4}\right)$	$\frac{\sqrt{\pi}}{a} \exp\left(\frac{s^2}{a^2}\right) \operatorname{erfc}\left(\frac{s}{a}\right)$, $a \geq 0$
33	$\left(t^2 - a^2\right)^{-\frac{1}{2}} H(t-a)$	$K_0(as), \quad a > 0$
34	$\delta^{(n)}(t-a), \quad n = 0, 1, \ldots$	$s^n \exp(-as)$

	$f(t)$	$\overline{f}(s)=\int_0^\infty \exp(-st)\, f(t)\, dt$
35	$t^{m\alpha+\beta-1}E^{(m)}_{\alpha,\beta}(\mp at)$, $m=0,1,2,\ldots$	$\frac{m!\, s^{\alpha-\beta}}{\left(s^\alpha \mp a\right)^{m+1}}$
36	$\frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{t}{2a}\right)^\nu J_\nu(at)$	$\left(s^2+a^2\right)^{-\left(\nu+\frac{1}{2}\right)}$, $\operatorname{Re}\nu > -\frac{1}{2}$
37	$\frac{1}{2}e^{-ct}\Big[\exp\left(-a\sqrt{b-c}\right)\times\operatorname{erfc}\left\{\frac{a}{\sqrt{4t}}-\sqrt{(b-c)\,t}\right\}-\exp\left(a\sqrt{b-c}\right)\times\operatorname{erfc}\left\{\frac{a}{\sqrt{4t}}+\sqrt{(b-c)\,t}\right\}\Big]$	$\frac{\exp\left(-a\sqrt{s+b}\right)}{(s+c)\sqrt{(s+b)}}$
38	$\frac{1}{2}e^{-ct}\Big[\exp\left(-a\sqrt{b-c}\right)\times\operatorname{erfc}\left\{\frac{a}{\sqrt{4t}}-t\sqrt{b-c}\right\}-\exp\left(a\sqrt{b-c}\right)\times\operatorname{erfc}\left\{\frac{a}{\sqrt{4t}}+t\sqrt{b-c}\right\}\Big]$	$\frac{\exp\left(-a\sqrt{s+b}\right)}{(s+c)}$
39	$e^{-bt}\left[\sqrt{\frac{4t}{\pi}}\exp\left(-\frac{a^2}{4t}\right)-a\operatorname{erfc}\left(\frac{a}{\sqrt{4t}}\right)\right]$	$\frac{\exp\left(-a\sqrt{s+b}\right)}{(s+b)^{3/2}}$
40	$e^{-bt}\left[\left(t+\frac{1}{2}a^2\right)\operatorname{erfc}\left(\frac{a}{\sqrt{4t}}\right)-\sqrt{\frac{ta^2}{\pi}}\exp\left(-\frac{a^2}{4t}\right)\right]$	$\frac{\exp\left(-a\sqrt{s+b}\right)}{(s+b)^2}$