We have, from (6.13.12),

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos kx\,dk \int_0^1 \cos kt\,dt = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin k}{k}\right)\cos kx\,dk,$$

or,

$$1 = \frac{2}{\pi}\int_0^\infty \frac{\sin k}{k}\cos kx\,dk.$$

6.14 Exercises

1. Find the Fourier series of the following functions:

   (a) $f(x) = \begin{cases} x & -\pi < x < 0 \\ h & 0 < x < \pi, \end{cases}$ h is a constant

   (b) $f(x) = \begin{cases} 1 & -\pi < x < 0 \\ x^2 & 0 < x < \pi, \end{cases}$

   (c) $f(x) = x + \sin x \quad -\pi < x < \pi,$

   (d) $f(x) = 1 + x \quad -\pi < x < \pi,$

   (e) $f(x) = e^x \quad -\pi < x < \pi,$

   (f) $f(x) = 1 + x + x^2 \quad -\pi < x < \pi.$

2. Determine the Fourier sine series of the following functions:

   (a) $f(x) = \pi - x \quad 0 < x < \pi,$

   (b) $f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 2 & \pi/2 < x < \pi, \end{cases}$

   (c) $f(x) = x^2 \quad 0 < x < \pi,$

   (d) $f(x) = \cos x \quad 0 < x < \pi,$

   (e) $f(x) = x^3 \quad 0 < x < \pi,$

   (f) $f(x) = e^x \quad 0 < x < \pi.$

3. Obtain the Fourier cosine series representation for the following functions:

   (a) $f(x) = \pi + x \qquad 0 < x < \pi,$

   (b) $f(x) = x \qquad 0 < x < \pi,$

   (c) $f(x) = x^2 \qquad 0 < x < \pi,$

   (d) $f(x) = \sin 3x \qquad 0 < x < \pi,$

   (e) $f(x) = e^x \qquad 0 < x < \pi,$

   (f) $f(x) = \cosh x \qquad 0 < x < \pi.$

4. Expand the following functions in a Fourier series:

   (a) $f(x) = x^2 + x \qquad -1 < x < 1,$

   (b) $f(x) = \begin{cases} 1 & 0 < x < 3 \\ 0 & 3 < x < 6, \end{cases}$

   (c) $f(x) = \sin(\pi x/l) \qquad 0 < x < l,$

   (d) $f(x) = x^3 \qquad -2 < x < 2,$

   (e) $f(x) = e^{-x} \qquad 0 < x < 1,$

   (f) $f(x) = \sinh x \qquad -1 < x < 1.$

5. Expand the following functions in a complex Fourier series:

   (a) $f(x) = e^{2x} \qquad -\pi < x < \pi,$

   (b) $f(x) = \cosh x \qquad -\pi < x < \pi,$

   (c) $f(x) = \begin{cases} 1 & -\pi < x < 0 \\ \cos x & 0 < x < \pi, \end{cases}$

   (d) $f(x) = x \qquad -1 < x < 1,$

   (e) $f(x) = x^2 \qquad -\pi < x < \pi,$

   (f) $f(x) = \sinh(\pi x/2) \qquad -2 < x < 2.$

6. (a) Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x/2, & 0 < x < \pi. \end{cases}$$

(b) With the use of the Fourier series of $f(x)$ in 6(a), show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

7. (a) Determine the Fourier series of the function

$$f(x) = x^2, \qquad -l < x < l.$$

(b) With the use of the Fourier series of $f(x)$ in 7(a), show that

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

8. Determine the Fourier series expansion of each of the following functions by performing the differentiation of the appropriate Fourier series:

(a) $\sin^2 x \qquad 0 < x < \pi,$

(b) $\cos^2 x \qquad 0 < x < \pi,$

(c) $\sin x \cos x \qquad 0 < x < \pi,$

(d) $\cos x + \cos 2x \qquad 0 < x < \pi,$

(e) $\cos x + \cos 2x \qquad 0 < x < \pi.$

9. Find the functions represented by the new series which are obtained by termwise integration of the following series from 0 to x:

(a) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx = x/2 \qquad -\pi < x < \pi,$

(b) $\frac{3}{2} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1-(-1)^k}{k} \sin kx = \begin{cases} 1 & -\pi < x < 0 \\ 2 & 0 < x < \pi, \end{cases}$

(c) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\cos kx}{k} = \ln\left(2 \cos \frac{x}{2}\right) \qquad -\pi < x < \pi,$