The complex conjugate is important because it permits us to switch from complex to real. Indeed, by multiplication, $z\bar{z} = x^2 + y^2$ (verify!). By addition and subtraction, $z + \bar{z} = 2x$, $z - \bar{z} = 2iy$. We thus obtain for the real part x and the imaginary part y (not iy!) of $z = x + iy$ the important formulas

$$\text{(8)} \qquad \operatorname{Re} z = x = \frac{1}{2}(z + \bar{z}), \qquad \operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z}).$$

If z is real, $z = x$, then $\bar{z} = z$ by the definition of $\bar{z}$, and conversely. Working with conjugates is easy, since we have

$$\text{(9)} \qquad \begin{aligned} \overline{(z_1 + z_2)} &= \bar{z}_1 + \bar{z}_2, & \overline{(z_1 - z_2)} &= \bar{z}_1 - \bar{z}_2, \\ \overline{(z_1 z_2)} &= \bar{z}_1 \bar{z}_2, & \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\bar{z}_1}{\bar{z}_2}. \end{aligned}$$

EXAMPLE 3 Illustration of (8) and (9)

Let $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$. Then by (8),

$$\operatorname{Im} z_1 = \frac{1}{2i}[(4 + 3i) - (4 - 3i)] = \frac{3i + 3i}{2i} = 3.$$

Also, the multiplication formula in (9) is verified by

$$\begin{aligned} \overline{(z_1 z_2)} &= \overline{(4 + 3i)(2 + 5i)} = \overline{(-7 + 26i)} = -7 - 26i, \\ \bar{z}_1 \bar{z}_2 &= (4 - 3i)(2 - 5i) = -7 - 26i. \end{aligned}$$

■

PROBLEM SET 13.1

1. **(Powers of i)** Show that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \cdots$ and $1/i = -i$, $1/i^2 = -1$, $1/i^3 = i, \cdots$.
2. **(Rotation)** Multiplication by i is geometrically a counterclockwise rotation through $\pi/2$ (90°). Verify this by graphing z and iz and the angle of rotation for $z = 2 + 2i$, $z = -1 - 5i$, $z = 4 - 3i$.
3. **(Division)** Verify the calculation in (7).
4. **(Multiplication)** If the product of two complex numbers is zero, show that at least one factor must be zero.
5. Show that $z = x + iy$ is pure imaginary if and only if $\bar{z} = -z$.
6. **(Laws for conjugates)** Verify (9) for $z_1 = 24 + 10i$, $z_2 = 4 + 6i$.

7–15 COMPLEX ARITHMETIC

Let $z_1 = 2 + 3i$ and $z_2 = 4 - 5i$. Showing the details of your work, find (in the form $x + iy$):

7. $(5z_1 + 3z_2)^2$
8. $\bar{z}_1 \bar{z}_2$
9. $\operatorname{Re}(1/z_1^{\,2})$
10. $\operatorname{Re}(z_2^{\,2})$, $(\operatorname{Re} z_2)^2$
11. z_2/z_1
12. $\bar{z}_1/\bar{z}_2$, $\overline{(z_1/z_2)}$
13. $(4z_1 - z_2)^2$
14. $\bar{z}_1/z_1$, $z_1/\bar{z}_1$
15. $(z_1 + z_2)/(z_1 - z_2)$

16–19 Let $z = x + iy$. Find:

16. $\operatorname{Im} z^3$, $(\operatorname{Im} z)^3$
17. $\operatorname{Re}(1/\bar{z})$
18. $\operatorname{Im}[(1 + i)^8 z^2]$
19. $\operatorname{Re}(1/\bar{z}^2)$
20. **(Laws of addition and multiplication)** Derive the following laws for complex numbers from the corresponding laws for real numbers.

$$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1 \quad \textit{(Commutative laws)}$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad \textit{(Associative laws)}$$

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \qquad \textit{(Distributive law)}$$

$$0 + z = z + 0 = z,$$

$$z + (-z) = (-z) + z = 0, \qquad z \cdot 1 = z.$$

hence the w corresponding to $k = 0$, etc. Consequently, $\sqrt[n]{z}$, for $z \neq 0$, has the n distinct values

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) \tag{15}$$

where $k = 0, 1, \cdots, n - 1$. These n values lie on a circle of radius $\sqrt[n]{r}$ with center at the origin and constitute the vertices of a regular polygon of n sides. The value of $\sqrt[n]{z}$ obtained by taking the principal value of arg z and $k = 0$ in (15) is called the **principal value** of $w = \sqrt[n]{z}$.

Taking $z = 1$ in (15), we have $|z| = r = 1$ and Arg $z = 0$. Then (15) gives

$$\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \qquad k = 0, 1, \cdots, n - 1. \tag{16}$$

These n values are called the ***n*th roots of unity.** They lie on the circle of radius 1 and center 0, briefly called the **unit circle** (and used quite frequently!). Figures 324–326 show $\sqrt[3]{1} = 1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\,i$, $\sqrt[4]{1} = \pm 1, \pm i$, and $\sqrt[5]{1}$.

If ω denotes the value corresponding to $k = 1$ in (16), then the n values of $\sqrt[n]{1}$ can be written as

$$1, \omega, \omega^2, \cdots, \omega^{n-1}.$$

More generally, if w_1 is any nth root of an arbitrary complex number z ($\neq 0$), then the n values of $\sqrt[n]{z}$ in (15) are

$$w_1, \qquad w_1\omega, \qquad w_1\omega^2, \qquad \cdots, \qquad w_1\omega^{n-1} \tag{17}$$

because multiplying w_1 by ω^k corresponds to increasing the argument of w_1 by $2k\pi/n$. Formula (17) motivates the introduction of roots of unity and shows their usefulness.

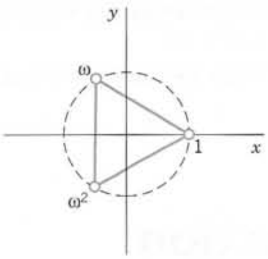

Fig. 324. $\sqrt[3]{1}$

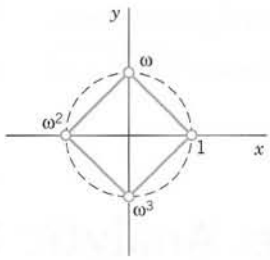

Fig. 325. $\sqrt[4]{1}$

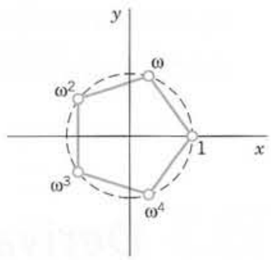

Fig. 326. $\sqrt[5]{1}$

PROBLEM SET 13.2

1–8 POLAR FORM

Do these problems very carefully since polar forms will be needed frequently. Represent in polar form and graph in the complex plane as in Fig. 322 on p. 608. (Show the details of your work.)

1. $3 - 3i$

2. $2i, -2i$

3. -5

4. $\frac{1}{2} + \frac{1}{4}\pi i$

5. $\dfrac{1 + i}{1 - i}$

6. $\dfrac{3\sqrt{2} + 2i}{-\sqrt{2} - (2/3)i}$

7. $\dfrac{-6 + 5i}{3i}$ **8.** $\dfrac{2 + 3i}{5 + 4i}$

9–15 PRINCIPAL ARGUMENT

Determine the principal value of the argument.

9. $-1 - i$ **10.** $-20 + i, -20 - i$

11. $4 \pm 3i$ **12.** $-\pi^2$

13. $7 \pm 7i$ **14.** $(1 + i)^{12}$

15. $(9 + 9i)^3$

16–20 CONVERSION TO $x + iy$

Represent in the form $x + iy$ and graph it in the complex plane.

16. $\cos \frac{1}{2}\pi + i \sin (\pm\frac{1}{2}\pi)$ **17.** $3(\cos 0.2 + i \sin 0.2)$

18. $4(\cos \frac{1}{3}\pi \pm i \sin \frac{1}{3}\pi)$ **19.** $\cos(-1) + i \sin(-1)$

20. $12(\cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi)$

21–25 ROOTS

Find and graph all roots in the complex plane.

21. $\sqrt{-i}$ **22.** $\sqrt[8]{1}$

23. $\sqrt[4]{-1}$ **24.** $\sqrt[3]{3 + 4i}$

25. $\sqrt[5]{-1}$

26. TEAM PROJECT. Square Root. (a) Show that $w = \sqrt{z}$ has the values

$$w_1 = \sqrt{r}\left[\cos\frac{\theta}{2} + i \sin\frac{\theta}{2}\right],$$

$$(18) \quad w_2 = \sqrt{r}\left[\cos\left(\frac{\theta}{2} + \pi\right) + i \sin\left(\frac{\theta}{2} + \pi\right)\right] = -w_1.$$

(b) Obtain from (18) the often more practical formula

$$(19) \quad \sqrt{z} = \pm\left[\sqrt{\tfrac{1}{2}(|z| + x)} + (\text{sign } y)\, i \sqrt{\tfrac{1}{2}(|z| - x)}\right]$$

where sign $y = 1$ if $y \geqq 0$, sign $y = -1$ if $y < 0$, and all square roots of positive numbers are taken with positive sign. *Hint:* Use (10) in App. A3.1 with $x = \theta/2$.

(c) Find the square roots of $4i$, $16 - 30i$, and $9 + 8\sqrt{7}i$ by both (18) and (19) and comment on the work involved.

(d) Do some further examples of your own and apply a method of checking your results.

27–30 EQUATIONS

Solve and graph all solutions, showing the details:

27. $z^2 - (8 - 5i)z + 40 - 20i = 0$ (Use (19).)

28. $z^4 + (5 - 14i)z^2 - (24 + 10i) = 0$

29. $8z^2 - (36 - 6i)z + 42 - 11i = 0$

30. $z^4 + 16 = 0$. Then use the solutions to factor $z^4 + 16$ into quadratic factors with *real* coefficients.

31. CAS PROJECT. Roots of Unity and Their Graphs. Write a program for calculating these roots and for graphing them as points on the unit circle. Apply the program to $z^n = 1$ with $n = 2, 3, \cdots, 10$. Then extend the program to one for arbitrary roots, using an idea near the end of the text, and apply the program to examples of your choice.

32–35 INEQUALITIES AND AN EQUATION

Verify or prove as indicated.

32. (Re and Im) Prove $|\text{Re } z| \leqq |z|$, $|\text{Im } z| \leqq |z|$.

33. (Parallelogram equality) Prove

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

Explain the name.

34. (Triangle inequality) Verify (6) for $z_1 = 4 + 7i$, $z_2 = 5 + 2i$.

35. (Triangle inequality) Prove (6).

13.3 Derivative. Analytic Function

Our study of complex functions will involve point sets in the complex plane. Most important will be the following ones.

Circles and Disks. Half-Planes

The **unit circle** $|z| = 1$ (Fig. 327) has already occurred in Sec. 13.2. Figure 328 shows a general circle of radius ρ and center a. Its equation is

$$|z - a| = \rho$$

Surprising as Example 4 may be, it merely illustrates that differentiability of a *complex* function is a rather severe requirement.

The idea of proof (approach of z from different directions) is basic and will be used again as the crucial argument in the next section.

Analytic Functions

Complex analysis is concerned with the theory and application of "analytic functions," that is, functions that are differentiable in some domain, so that we can do "calculus in complex." The definition is as follows.

DEFINITION

Analyticity

A function $f(z)$ is said to be *analytic in a domain D* if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be *analytic at a point* $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0.

Also, by an **analytic function** we mean a function that is analytic in *some* domain.

Hence analyticity of $f(z)$ at z_0 means that $f(z)$ has a derivative at every point in some neighborhood of z_0 (including z_0 itself since, by definition, z_0 is a point of all its neighborhoods). This concept is *motivated* by the fact that it is of no practical interest if a function is differentiable merely at a single point z_0 but not throughout some neighborhood of z_0. Team Project 26 gives an example.

A more modern term for *analytic in D* is *holomorphic in D.*

EXAMPLE 5 Polynomials, Rational Functions

The nonnegative integer powers $1, z, z^2, \cdots$ are analytic in the entire complex plane, and so are **polynomials,** that is, functions of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$$

where $c_0, \cdots, c_n$ are complex constants.

The quotient of two polynomials $g(z)$ and $h(z)$,

$$f(z) = \frac{g(z)}{h(z)},$$

is called a **rational function.** This f is analytic except at the points where $h(z) = 0$; here we assume that common factors of g and h have been canceled.

Many further analytic functions will be considered in the next sections and chapters. ■

The concepts discussed in this section extend familiar concepts of calculus. Most important is the concept of an analytic function, the exclusive concern of complex analysis. Although many simple functions are not analytic, the large variety of remaining functions will yield a most beautiful branch of mathematics that is very useful in engineering and physics.

PROBLEM SET 13.3

1–10 CURVES AND REGIONS OF PRACTICAL INTEREST

Find and sketch or graph the sets in the complex plane given by

1. $|z - 3 - 2i| = \frac{4}{3}$

2. $1 \leqq |z - 1 + 4i| \leqq 5$

3. $0 < |z - 1| < 1$

4. $-\pi < \text{Re}\, z < \pi$

5. $\text{Im}\, z^2 = 2$

6. $\text{Re}\, z > -1$

7. $|z + 1| = |z - 1|$

8. $|\text{Arg}\, z| \leqq \frac{1}{4}\pi$

9. $\text{Re}\, z \leqq \text{Im}\, z$

10. $\text{Re}\,(1/z) < 1$

11. WRITING PROJECT. Sets in the Complex Plane. Extend the part of the text on sets in the complex plane by formulating that part in your own words and including examples of your own and comparing with calculus when applicable.

COMPLEX FUNCTIONS AND DERIVATIVES

12–15 **Function Values.** Find Re f and Im f. Also find their values at the given point z.

12. $f = 3z^2 - 6z + 3i,\ z = 2 + i$

13. $f = z/(z + 1),\ z = 4 - 5i$

14. $f = 1/(1 - z),\ z = \frac{1}{2} + \frac{1}{4}i$

15. $f = 1/z^2,\ z = 1 + i$

16–19 **Continuity.** Find out (and give reason) whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function f is equal to:

16. $[\text{Re}\,(z^2)]/|z|^2$

17. $[\text{Im}\,(z^2)]/|z|$

18. $|z|^2\ \text{Re}\,(1/z)$

19. $(\text{Im}\, z)/(1 - |z|)$

20–24 **Derivative.** Differentiate

20. $(z^2 - 9)/(z^2 + 1)$

21. $(z^3 + i)^2$

22. $(3z + 4i)/(1.5iz - 2)$

23. $i/(1 - z)^2$

24. $z^2/(z + i)^2$

25. CAS PROJECT. Graphing Functions. Find and graph Re f, Im f, and $|f|$ as surfaces over the z-plane. Also graph the two families of curves Re $f(z) = const$ and Im $f(z) = const$ in the same figure, and the curves $|f(z)| = const$ in another figure, where (a) $f(z) = z^2$, (b) $f(z) = 1/z$, (c) $f(z) = z^4$.

26. TEAM PROJECT. Limit, Continuity, Derivative

(a) Limit. Prove that (1) is equivalent to the pair of relations

$$\lim_{z \to z_0} \text{Re}\, f(z) = \text{Re}\, l, \qquad \lim_{z \to z_0} \text{Im}\, f(z) = \text{Im}\, l.$$

(b) Limit. If $\lim_{z \to z_0} f(z)$ exists, show that this limit is unique.

(c) Continuity. If $z_1, z_2, \cdots$ are complex numbers for which $\lim_{n\to\infty} z_n = a$, and if $f(z)$ is continuous at $z = a$, show that $\lim_{n\to\infty} f(z_n) = f(a)$.

(d) Continuity. If $f(z)$ is differentiable at z_0, show that $f(z)$ is continuous at z_0.

(e) Differentiability. Show that $f(z) = \text{Re}\, z = x$ is not differentiable at any z. Can you find other such functions?

(f) Differentiability. Show that $f(z) = |z|^2$ is differentiable only at $z = 0$; hence it is nowhere analytic.

13.4 Cauchy–Riemann Equations. Laplace's Equation

The Cauchy–Riemann equations are the most important equations in this chapter and one of the pillars on which complex analysis rests. They provide a criterion (a test) for the analyticity of a complex function

$$w = f(z) = u(x, y) + iv(x, y).$$

Roughly, f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two **Cauchy–Riemann equations**[4]

$$u_x = v_y, \qquad u_y = -v_x \tag{1}$$

[4]The French mathematician AUGUSTIN-LOUIS CAUCHY (see Sec. 2.5) and the German mathematicians BERNHARD RIEMANN (1826–1866) and KARL WEIERSTRASS (1815–1897; see also Sec. 15.5) are the founders of complex analysis. Riemann received his Ph.D. (in 1851) under Gauss (Sec. 5.4) at Göttingen, where he also taught until he died, when he was only 39 years old. He introduced the concept of the integral as it is used in basic calculus courses, and made important contributions to differential equations, number theory, and mathematical physics. He also developed the so-called Riemannian geometry, which is the mathematical foundation of Einstein's theory of relativity; see Ref. [GR9] in App. 1.

Example 4 illustrates that *a conjugate of a given harmonic function is uniquely determined up to an arbitrary real additive constant.*

The Cauchy–Riemann equations are the most important equations in this chapter. Their relation to Laplace's equation opens wide ranges of engineering and physical applications, as we shall show in Chap. 18.

PROBLEM SET 13.4

1–10 CAUCHY–RIEMANN EQUATIONS

Are the following functions analytic? [Use (1) or (7).]

1. $f(z) = z^4$
2. $f(z) = \text{Im}\,(z^2)$
3. $e^{2x}(\cos y + i \sin y)$
4. $f(z) = 1/(1 - z^4)$
5. $e^{-x}(\cos y - i \sin y)$
6. $f(z) = \text{Arg}\,\pi z$
7. $f(z) = \text{Re}\,z + \text{Im}\,z$
8. $f(z) = \ln|z| + i\,\text{Arg}\,z$
9. $f(z) = i/z^8$
10. $f(z) = z^2 + 1/z^2$

11. (Cauchy–Riemann equations in polar form) Derive (7) from (1).

12–21 HARMONIC FUNCTIONS

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

12. $u = xy$
13. $v = xy$
14. $v = -y/(x^2 + y^2)$
15. $u = \ln|z|$
16. $v = \ln|z|$
17. $u = x^3 - 3xy^2$
18. $u = 1/(x^2 + y^2)$
19. $v = (x^2 - y^2)^2$
20. $u = \cos x \cosh y$
21. $u = e^{-x} \sin 2y$

22–24 Determine a, b, c such that the given functions are harmonic and find a harmonic conjugate.

22. $u = e^{3x} \cos ay$
23. $u = \sin x \cosh cy$
24. $u = ax^3 + by^3$

25. (Harmonic conjugate) Show that if u is harmonic and v is a harmonic conjugate of u, then u is a harmonic conjugate of $-v$.

26. TEAM PROJECT. Conditions for $f(z) = const.$ Let $f(z)$ be analytic. Prove that each of the following conditions is sufficient for $f(z) = const.$

(a) $\text{Re}\,f(z) = const$

(b) $\text{Im}\,f(z) = const$

(c) $f'(z) = 0$

(d) $|f(z)| = const$ (see Example 3)

27. (Two further formulas for the derivative). Formulas (4), (5), and (11) (below) are needed from time to time. Derive

$$(11) \quad f'(z) = u_x - iu_y, \qquad f'(z) = v_y + iv_x.$$

28. CAS PROJECT. Equipotential Lines. Write a program for graphing equipotential lines $u = const$ of a harmonic function u and of its conjugate v on the same axes. Apply the program to (a) $u = x^2 - y^2$, $v = 2xy$, (b) $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$, (c) $u = e^x \cos y$, $v = e^x \sin y$.

13.5 Exponential Function

In the remaining sections of this chapter we discuss the basic elementary complex functions, the exponential function, trigonometric functions, logarithm, and so on. They will be counterparts to the familiar functions of calculus, to which they reduce when $z = x$ is real. They are indispensable throughout applications, and some of them have interesting properties not shared by their real counterparts.

We begin with one of the most important analytic functions, the complex **exponential function**

$$e^z, \qquad \text{also written} \qquad \exp z.$$

The definition of e^z in terms of the real functions e^x, $\cos y$, and $\sin y$ is

$$(1) \qquad e^z = e^x(\cos y + i \sin y).$$

To solve the equation $e^z = 3 + 4i$, note first that $|e^z| = e^x = 5$, $x = \ln 5 = 1.609$ is the real part of all solutions. Now, since $e^x = 5$,

$$e^x \cos y = 3, \quad e^x \sin y = 4, \quad \cos y = 0.6, \quad \sin y = 0.8, \quad y = 0.927.$$

Ans. $z = 1.609 + 0.927i \pm 2n\pi i$ $(n = 0, 1, 2, \cdots)$. These are infinitely many solutions (due to the periodicity of e^z). They lie on the vertical line $x = 1.609$ at a distance 2π from their neighbors. ■

To summarize: many properties of $e^z = \exp z$ parallel those of e^x; an exception is the periodicity of e^z with $2\pi i$, which suggested the concept of a fundamental region. Keep in mind that e^z is an *entire function*. (Do you still remember what that means?)

PROBLEM SET 13.5

1. Using the Cauchy–Riemann equations, show that e^z is entire.

2–8 Values of e^z. Compute e^z in the form $u + iv$ and $|e^z|$, where z equals:

2. $3 + \pi i$
3. $1 + 2i$
4. $\sqrt{2} - \frac{1}{2}\pi i$
5. $7\pi i/2$
6. $(1 + i)\pi$
7. $0.8 - 5i$
8. $9\pi i/2$

9–12 Real and Imaginary Parts. Find Re and Im of:

9. e^{-2z}
10. e^{z^3}
11. e^{z^2}
12. $e^{1/z}$

13–17 Polar Form. Write in polar form:

13. $\sqrt{i}$
14. $1 + i$
15. $\sqrt[n]{z}$
16. $3 + 4i$
17. -9

18–21 Equations. Find all solutions and graph some of them in the complex plane.

18. $e^{3z} = 4$
19. $e^z = -2$
20. $e^z = 0$
21. $e^z = 4 - 3i$

22. **TEAM PROJECT. Further Properties of the Exponential Function. (a) Analyticity.** Show that e^z is entire. What about $e^{1/z}$? $e^{\bar{z}}$? $e^x(\cos ky + i \sin ky)$? (Use the Cauchy–Riemann equations.)

    (b) Special values. Find all z such that (i) e^z is real, (ii) $|e^{-z}| < 1$, (iii) $e^{\bar{z}} = \overline{e^z}$.

    (c) Harmonic function. Show that $u = e^{xy} \cos(x^2/2 - y^2/2)$ is harmonic and find a conjugate.

    (d) Uniqueness. It is interesting that $f(z) = e^z$ is uniquely determined by the two properties $f(x + i0) = e^x$ and $f'(z) = f(z)$, where f is assumed to be entire. Prove this using the Cauchy–Riemann equations.

13.6 Trigonometric and Hyperbolic Functions

Just as we extended the real e^x to the complex e^z in Sec. 13.5, we now want to extend the familiar *real* trigonometric functions to *complex trigonometric functions*. We can do this by the use of the Euler formulas (Sec. 13.5)

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x.$$

By addition and subtraction we obtain for the *real* cosine and sine

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

This suggests the following definitions for complex values $z = x + iy$:

(13)
$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z},$$
$$\operatorname{sech} z = \frac{1}{\cosh z}, \qquad \operatorname{csch} z = \frac{1}{\sinh z}.$$

Complex Trigonometric and Hyperbolic Functions Are Related. If in (11), we replace z by iz and then use (1), we obtain

(14) $$\cosh iz = \cos z, \qquad \sinh iz = i \sin z.$$

Similarly, if in (1) we replace z by iz and then use (11), we obtain conversely

(15) $$\cos iz = \cosh z, \qquad \sin iz = i \sinh z.$$

Here we have another case of *unrelated* real functions that have *related* complex analogs, pointing again to the advantage of working in complex in order to get both a more unified formalism and a deeper understanding of special functions. This is one of the main reasons for the importance of complex analysis to the engineer and physicist.

PROBLEM SET 13.6

1. Prove that $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ are entire functions.

2. Verify by differentiation that $\operatorname{Re} \cos z$ and $\operatorname{Im} \sin z$ are harmonic.

3–6 FORMULAS FOR HYPERBOLIC FUNCTIONS
Show that

3. $$\cosh z = \cosh x \cos y + i \sinh x \sin y$$
$$\sinh z = \sinh x \cos y + i \cosh x \sin y.$$

4. $$\cosh (z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$
$$\sinh (z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.$$

5. $\cosh^2 z - \sinh^2 z = 1$

6. $\cosh^2 z + \sinh^2 z = \cosh 2z$

7–15 Function Values. Compute (in the form $u + iv$)

7. $\cos (1 + i)$

8. $\sin (1 + i)$

9. $\sin 5i$, $\cos 5i$

10. $\cos 3\pi i$

11. $\cosh (-2 + 3i)$, $\cos (-3 - 2i)$

12. $-i \sinh (-\pi + 2i)$, $\sin (2 + \pi i)$

13. $\cosh (2n + 1)\pi i$, $n = 1, 2, \cdots$

14. $\sinh (4 - 3i)$

15. $\cosh (4 - 6\pi i)$

16. (Real and imaginary parts) Show that
$$\operatorname{Re} \tan z = \frac{\sin x \cos x}{\cos^2 x + \sinh^2 y},$$
$$\operatorname{Im} \tan z = \frac{\sinh y \cosh y}{\cos^2 x + \sinh^2 y}.$$

17–21 Equations. Find all solutions of the following equations.

17. $\cosh z = 0$

18. $\sin z = 100$

19. $\cos z = 2i$

20. $\cosh z = -1$

21. $\sinh z = 0$

22. Find all z for which (a) $\cos z$, (b) $\sin z$ has real values.

23–25 Equations and Inequalities. Using the definitions, prove:

23. $\cos z$ is even, $\cos (-z) = \cos z$, and $\sin z$ is odd, $\sin (-z) = -\sin z$.

24. $|\sinh y| \leqq |\cos z| \leqq \cosh y$, $|\sinh y| \leqq |\sin z| \leqq \cosh y$. Conclude that the complex cosine and sine are not bounded in the whole complex plane.

25. $\sin z_1 \cos z_2 = \frac{1}{2}[\sin (z_1 + z_2) + \sin (z_1 - z_2)]$

If $c = n = 1, 2, \cdots$, then z^n is single-valued and identical with the usual nth power of z. If $c = -1, -2, \cdots$, the situation is similar.

If $c = 1/n$, where $n = 2, 3, \cdots$, then

$$z^c = \sqrt[n]{z} = e^{(1/n) \ln z} \qquad (z \neq 0),$$

the exponent is determined up to multiples of $2\pi i/n$ and we obtain the n distinct values of the nth root, in agreement with the result in Sec. 13.2. If $c = p/q$, the quotient of two positive integers, the situation is similar, and z^c has only finitely many distinct values. However, if c is real irrational or genuinely complex, then z^c is infinitely many-valued.

EXAMPLE 3 General Power

$$i^i = e^{i \ln i} = \exp(i \ln i) = \exp\left[i\left(\frac{\pi}{2} i \pm 2n\pi i\right)\right] = e^{-(\pi/2) \mp 2n\pi}.$$

All these values are real, and the principal value ($n = 0$) is $e^{-\pi/2}$.

Similarly, by direct calculation and multiplying out in the exponent,

$$(1 + i)^{2-i} = \exp[(2 - i) \ln(1 + i)] = \exp[(2 - i)\{\ln \sqrt{2} + \tfrac{1}{4}\pi i \pm 2n\pi i\}]$$
$$= 2e^{\pi/4 \pm 2n\pi}[\sin(\tfrac{1}{2} \ln 2) + i \cos(\tfrac{1}{2} \ln 2)].$$

It is a ***convention*** that for real positive $z = x$ the expression z^c means $e^{c \ln x}$ where $\ln x$ is the elementary real natural logarithm (that is, the principal value $\text{Ln}\, z$ ($z = x > 0$) in the sense of our definition). Also, if $z = e$, the base of the natural logarithm, $z^c = e^c$ is *conventionally* regarded as the unique value obtained from (1) in Sec. 13.5.

From (7) we see that for any complex number a,

$$a^z = e^{z \ln a}. \qquad (8)$$

We have now introduced the complex functions needed in practical work, some of them (e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$) entire (Sec. 13.5), some of them ($\tan z$, $\cot z$, $\tanh z$, $\coth z$) analytic except at certain points, and one of them ($\ln z$) splitting up into infinitely many functions, each analytic except at 0 and on the negative real axis.

For the **inverse trigonometric** and **hyperbolic functions** see the problem set.

PROBLEM SET 13.7

1–9 Principal Value Ln z. Find $\text{Ln}\, z$ when z equals:

1. -10
2. $2 + 2i$
3. $2 - 2i$
4. $-5 \pm 0.1i$
5. $-3 - 4i$
6. -100
7. $0.6 + 0.8i$
8. $-ei$
9. $1 - i$

10–16 All Values of ln z. Find all values and graph some of them in the complex plane.

10. $\ln 1$
11. $\ln(-1)$
12. $\ln e$
13. $\ln(-6)$
14. $\ln(4 + 3i)$
15. $\ln(-e^{-i})$
16. $\ln(e^{3i})$
17. Show that the set of values of $\ln(i^2)$ differs from the set of values of $2 \ln i$.

18–21 Equations. Solve for z:

18. $\ln z = (2 - \tfrac{1}{2}i)\pi$
19. $\ln z = 0.3 + 0.7i$
20. $\ln z = e - \pi i$
21. $\ln z = 2 + \tfrac{1}{4}\pi i$

22–28 **General Powers.** Showing the details of your work, find the principal value of:

22. i^{2i}, $(2i)^i$

23. 4^{3+i}

24. $(1-i)^{1+i}$

25. $(1+i)^{1-i}$

26. $(-1)^{1-2i}$

27. $i^{1/2}$

28. $(3-4i)^{1/3}$

29. How can you find the answer to Prob. 24 from the answer to Prob. 25?

30. TEAM PROJECT. Inverse Trigonometric and Hyperbolic Functions. By definition, the **inverse sine** $w = \arcsin z$ is the relation such that $\sin w = z$. The **inverse cosine** $w = \arccos z$ is the relation such that $\cos w = z$. The **inverse tangent, inverse cotangent, inverse hyperbolic sine,** etc., are defined and denoted in a similar fashion. (Note that all these relations are ***multivalued.***) Using $\sin w = (e^{iw} - e^{-iw})/(2i)$ and similar representations of $\cos w$, etc., show that

(a) $\arccos z = -i \ln (z + \sqrt{z^2 - 1})$

(b) $\arcsin z = -i \ln (iz + \sqrt{1 - z^2})$

(c) $\operatorname{arccosh} z = \ln (z + \sqrt{z^2 - 1})$

(d) $\operatorname{arcsinh} z = \ln (z + \sqrt{z^2 + 1})$

(e) $\arctan z = \dfrac{i}{2} \ln \dfrac{i + z}{i - z}$

(f) $\operatorname{arctanh} z = \dfrac{1}{2} \ln \dfrac{1 + z}{1 - z}$

(g) Show that $w = \arcsin z$ is infinitely many-valued, and if w_1 is one of these values, the others are of the form $w_1 \pm 2n\pi$ and $\pi - w_1 \pm 2n\pi$, $n = 0, 1, \cdots$. (The *principal value of* $w = u + iv = \arcsin z$ is defined to be the value for which $-\pi/2 \leqq u \leqq \pi/2$ if $v \geqq 0$ and $-\pi/2 < u < \pi/2$ if $v < 0$.)

CHAPTER 13 REVIEW QUESTIONS AND PROBLEMS

1. Add, subtract, multiply, and divide $26 - 7i$ and $3 + 4i$ as well as their complex conjugates.

2. Write the two given numbers in Prob. 1 in polar form. Find the principal value of their arguments.

3. What is the triangle inequality? Its geometric meaning? Its significance?

4. If you know the values of $\sqrt[6]{1}$, how do you get from them the values of $\sqrt[6]{z}$ for any z?

5. State the definition of the derivative from memory. It looks similar to that in calculus. But what is the big difference?

6. What is an analytic function? How would you test for analyticity?

7. Can a function be differentiable at a point without being analytic there? If yes, give an example.

8. Are $|z|$, $\bar{z}$, $\operatorname{Re} z$, $\operatorname{Im} z$ analytic? Give reason.

9. State the definitions of e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ and the relations between these functions. Do these relations have analogs in real?

10. What properties of e^z are similar to those of e^x ? Which one is different?

11. What is the fundamental region of e^z ? Its significance?

12. What is an entire function? Give examples.

13. Why is $\ln z$ much more complicated than $\ln x$? Explain from memory.

14. What is the principal value of $\ln z$?

15. How is the general power z^c defined? Give examples.

16–21 **Complex Numbers.** Find, in the form $x + iy$, showing the details:

16. $(1 + i)^{12}$

17. $(-2 + 6i)^2$

18. $1/(3 - 7i)$

19. $(1 - i)/(1 + i)^2$

20. $\sqrt{-5 - 12i}$

21. $(43 - 19i)/(8 + i)$

22–26 **Polar Form.** Represent in polar form, with the principal argument:

22. $1 - 3i$

23. $-6 + 6i$

24. $\sqrt{20}/(4 + 2i)$

25. $-12i$

26. $2 + 2i$

27–30 **Roots.** Find and graph all values of

27. $\sqrt{8i}$

28. $\sqrt[4]{256}$

29. $\sqrt[4]{-1}$

30. $\sqrt{32 - 24i}$

31–35 **Analytic Functions.** Find $f(z) = u(x, y) + iv(x, y)$ with u or v as given. Check for analyticity.

31. $u = x/(x^2 + y^2)$

32. $v = e^{-3x} \sin 3y$

33. $u = x^2 - 2xy - y^2$

34. $u = \cos 2x \cosh 2y$

35. $v = e^{x^2 - y^2} \sin 2xy$

36–39 **Harmonic Functions.** Are the following functions harmonic? If so, find a harmonic conjugate.

36. x^2y^2

37. xy

38. $e^{-x/2} \cos \frac{1}{2}y$

39. $x^2 + y^2$

40–45 **Special Function Values.** Find the values of

40. $\sin (3 + 4\pi i)$

41. $\sinh 4\pi i$

42. $\cos (5\pi + 2i)$

43. $\operatorname{Ln} (0.8 + 0.6i)$

44. $\tan (1 + i)$

45. $\cosh (1 + \pi i)$

EXAMPLE 6 A Geometric Application

Geometric problems may also lead to initial value problems. For instance, find the curve through the point (1, 1) in the xy-plane having at each of its points the slope $-y/x$.

Solution. The slope y' should equal $-y/x$. This gives the ODE $y' = -y/x$. Its general solution is $y = c/x$ (see Example 1). This is a family of hyperbolas with the coordinate axes as asymptotes.

Now, for the curve to pass through (1, 1), we must have $y = 1$ when $x = 1$. Hence the initial condition is $y(1) = 1$. From this condition and $y = c/x$ we get $y(1) = c/1 = 1$; that is, $c = 1$. This gives the particular solution $y = 1/x$ (drawn somewhat thicker in Fig. 5). ■

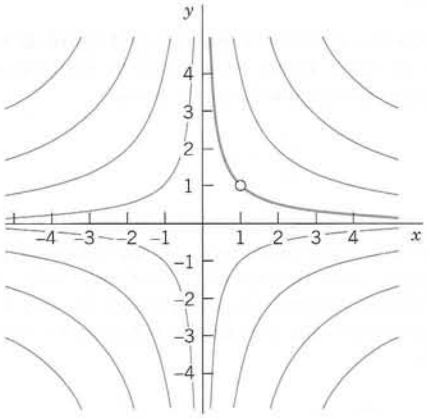

Fig. 5. Solutions of $y' = -y/x$ (hyperbolas)

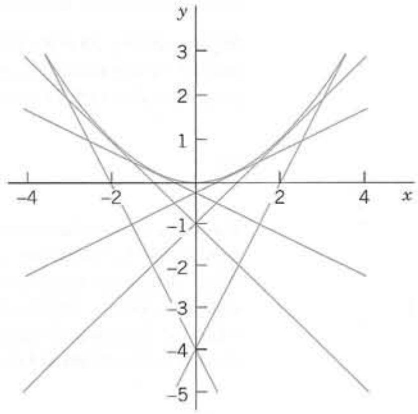

Fig. 6. Particular solutions and singular solution in Problem 16

PROBLEM SET 1.1

1–4 CALCULUS

Solve the ODE by integration.

1. $y' = -\sin \pi x$

2. $y' = e^{-3x}$

3. $y' = xe^{x^2/2}$

4. $y' = \cosh 4x$

5–9 VERIFICATION OF SOLUTION

State the order of the ODE. Verify that the given function is a solution. (a, b, c are arbitrary constants.)

5. $y' = 1 + y^2, \quad y = \tan (x + c)$

6. $y'' + \pi^2 y = 0, \quad y = a \cos \pi x + b \sin \pi x$

7. $y'' + 2y' + 10y = 0, \quad y = 4e^{-x} \sin 3x$

8. $y' + 2y = 4(x + 1)^2, \quad y = 5e^{-2x} + 2x^2 + 2x + 1$

9. $y''' = \cos x, \quad y = -\sin x + ax^2 + bx + c$

10–14 INITIAL VALUE PROBLEMS

Verify that y is a solution of the ODE. Determine from y the particular solution satisfying the given initial condition. Sketch or graph this solution.

10. $y' = 0.5y, \quad y = ce^{0.5x}, \quad y(2) = 2$

11. $y' = 1 + 4y^2, \quad y = \frac{1}{2} \tan (2x + c), \quad y(0) = 0$

12. $y' = y - x, \quad y = ce^x + x + 1, \quad y(0) = 3$

13. $y' + 2xy = 0, \quad y = ce^{-x^2}, \quad y(1) = 1/e$

14. $y' = y \tan x, \quad y = c \sec x, \quad y(0) = \frac{1}{2}\pi$

15. (Existence) (A) Does the ODE $y'^2 = -1$ have a (real) solution?

(B) Does the ODE $|y'| + |y| = 0$ have a general solution?

16. (Singular solution) An ODE may sometimes have an additional solution that cannot be obtained from the general solution and is then called a *singular solution.* The ODE $y'^2 - xy' + y = 0$ is of the kind. Show by differentiation and substitution that it has the general solution $y = cx - c^2$ and the singular solution $y = x^2/4$. Explain Fig. 6.

17–22 MODELING, APPLICATIONS

The following problems will give you a first impression of modeling. Many more problems on modeling follow throughout this chapter.

17. (Falling body) If we drop a stone, we can assume air resistance ("drag") to be negligible. Experiments show that under that assumption the acceleration $y'' = d^2y/dt^2$ of this motion is constant (equal to the so-called acceleration of gravity $g = 9.80 \text{ m/sec}^2 = 32 \text{ ft/sec}^2$). State this as an ODE for $y(t)$, the distance fallen as a function of time t. Solve the ODE to get the familiar law of free fall, $y = gt^2/2$.

18. (Falling body) If in Prob. 17 the stone starts at $t = 0$ from initial position y_0 with initial velocity $v = v_0$, show that the solution is $y = gt^2/2 + v_0 t + y_0$. How long does a fall of 100 m take if the body falls from rest? A fall of 200 m? (Guess first.)

19. (Airplane takeoff) If an airplane has a run of 3 km, starts with a speed 6 m/sec, moves with constant acceleration, and makes the run in 1 min, with what speed does it take off?

20. (Subsonic flight) The efficiency of the engines of subsonic airplanes depends on air pressure and usually is maximum near about 36 000 ft. Find the air pressure $y(x)$ at this height without calculation. *Physical information.* The rate of change $y'(x)$ is proportional to the pressure, and at 18 000 ft the pressure has decreased to half its value y_0 at sea level.

21. (Half-life) The half-life of a radioactive substance is the time in which half of the given amount disappears. Hence it measures the rapidity of the decay. What is the half-life of radium ${}_{88}\mathrm{Ra}^{226}$ (in years) in Example 5?

22. (Interest rates) Show by algebra that the investment $y(t)$ from a deposit y_0 after t years at an interest rate r is

$$y_a(t) = y_0[1 + r]^t \quad \text{(Interest compounded annually)}$$

$$y_d(t) = y_0[1 + (r/365)]^{365t} \quad \text{(Interest compounded daily).}$$

Recall from calculus that

$$[1 + (1/n)]^n \to e \text{ as } n \to \infty;$$

hence $[1 + (r/n)]^{nt} \to e^{rt}$; thus

$$y_c(t) = y_0 e^{rt} \quad \text{(Interest compounded continuously).}$$

What ODE does the last function satisfy? Let the initial investment be \$1000 and $r = 6\%$. Compute the value of the investment after 1 year and after 5 years using each of the three formulas. Is there much difference?

1.2 Geometric Meaning of $y' = f(x, y)$. Direction Fields

A first-order ODE

$$y' = f(x, y) \tag{1}$$

has a simple geometric interpretation. From calculus you know that the derivative $y'(x)$ of $y(x)$ is the slope of $y(x)$. Hence a solution curve of (1) that passes through a point (x_0, y_0) must have at that point the slope $y'(x_0)$ equal to the value of f at that point; that is,

$$y'(x_0) = f(x_0, y_0).$$

Read this paragraph again before you go on, and think about it.

It follows that you can indicate directions of solution curves of (1) by drawing short straight-line segments (*lineal elements*) in the xy-plane (as in Fig. 7a) and then fitting (approximate) solution curves through the direction field (or *slope field*) thus obtained. This method is important for two reasons.

1. You need not solve (1). This is essential because many ODEs have complicated solution formulas or none at all.
2. The method shows, in graphical form, the whole family of solutions and their typical properties. The accuracy is somewhat limited, but in most cases this does not matter.

Let us illustrate this method for the ODE

$$y' = xy. \tag{2}$$

PROBLEM SET 1.3

1. **(Constant of integration)** An arbitrary constant of integration must be introduced immediately when the integration is performed. Why is this important? Give an example of your own.

2–9 GENERAL SOLUTION

Find a general solution. Show the steps of derivation. Check your answer by substitution.

2. $y' + (x + 2)y^2 = 0$
3. $y' = 2 \sec 2y$
4. $y' = (y + 9x)^2 \quad (y + 9x = v)$
5. $yy' + 36x = 0$
6. $y' = (4x^2 + y^2)/(xy)$
7. $y' \sin \pi x = y \cos \pi x$
8. $xy' = \frac{1}{2}y^2 + y$
9. $y'e^{\pi x} = y^2 + 1$

10–19 INITIAL VALUE PROBLEMS

Find the particular solution. Show the steps of derivation, beginning with the general solution. (L, R, b are constants.)

10. $yy' + 4x = 0, \ y(0) = 3$
11. $dr/dt = -2tr, \ r(0) = r_0$
12. $2xyy' = 3y^2 + x^2, \ y(1) = 2$
13. $L \, dI/dt + RI = 0, \ I(0) = I_0$
14. $y' = y/x + (2x^3/y) \cos(x^2), \ y(\sqrt{\pi/2}) = \sqrt{\pi}$
15. $e^{2x}y' = 2(x + 2)y^3, \ y(0) = 1/\sqrt{5} \approx 0.45$
16. $xy' = y + 4x^5 \cos^2(y/x), \ y(2) = 0$
17. $y'x \ln x = y, \ y(3) = \ln 81$
18. $dr/d\theta = b[(dr/d\theta) \cos \theta + r \sin \theta], \ r(\frac{1}{2}\pi) = \pi, \ 0 < b < 1$
19. $yy' = (x - 1)e^{-y^2}, \ y(0) = 1$

20. **(Particular solution)** Introduce limits of integration in (3) such that y obtained from (3) satisfies the initial condition $y(x_0) = y_0$. Try the formula out on Prob. 19.

21–36 APPLICATIONS, MODELING

21. **(Curves)** Find all curves in the xy-plane whose tangents all pass through a given point (a, b).
22. **(Curves)** Show that any (nonvertical) straight line through the origin of the xy-plane intersects all solution curves of $y' = g(y/x)$ at the same angle.
23. **(Exponential growth)** If the growth rate of the amount of yeast at any time t is proportional to the amount present at that time and doubles in 1 week, how much yeast can be expected after 2 weeks? After 4 weeks?
24. **(Population model)** If in a population of bacteria the birth rate and death rate are proportional to the number of individuals present, what is the population as a function of time? Figure out the limiting situation for increasing time and interpret it.
25. **(Radiocarbon dating)** If a fossilized tree is claimed to be 4000 years old, what should be its ${}_6C^{14}$ content expressed as a percent of the ratio of ${}_6C^{14}$ to ${}_6C^{12}$ in a living organism?
26. **(Gompertz growth in tumors)** The Gompertz model is $y' = -Ay \ln y \ (A > 0)$, where $y(t)$ is the mass of tumor cells at time t. The model agrees well with clinical observations. The declining growth rate with increasing $y > 1$ corresponds to the fact that cells in the interior of a tumor may die because of insufficient oxygen and nutrients. Use the ODE to discuss the growth and decline of solutions (tumors) and to find constant solutions. Then solve the ODE.
27. **(Dryer)** If wet laundry loses half of its moisture during the first 5 minutes of drying in a dryer and if the rate of loss of moisture is proportional to the moisture content, when will the laundry be practically dry, say, when will it have lost 95% of its moisture? First guess.
28. **(Alibi?)** Jack, arrested when leaving a bar, claims that he has been inside for at least half an hour (which would provide him with an alibi). The police check the water temperature of his car (parked near the entrance of the bar) at the instant of arrest and again 30 minutes later, obtaining the values 190°F and 110°F, respectively. Do these results give Jack an alibi? (Solve by inspection.)
29. **(Law of cooling)** A thermometer, reading 10°C, is brought into a room whose temperature is 23°C. Two minutes later the thermometer reading is 18°C. How long will it take until the reading is practically 23°C, say, 22.8°C? First guess.
30. **(Torricelli's law)** How does the answer in Example 5 (the time when the tank is empty) change if the diameter of the hole is doubled? First guess.
31. **(Torricelli's law)** Show that (7) looks reasonable inasmuch as $\sqrt{2gh(t)}$ is the speed a body gains if it falls a distance h (and air resistance is neglected).
32. **(Rope)** To tie a boat in a harbor, how many times must a rope be wound around a bollard (a vertical rough cylindrical post fixed on the ground) so that a man holding one end of the rope can resist a force exerted by the boat one thousand times greater than the man can exert? First guess. Experiments show that the change ΔS of the force S in a small portion of the rope is proportional to S and to the small angle $\Delta\phi$ in Fig. 13. Take the proportionality constant 0.15.

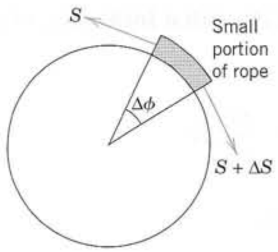

Fig. 13. Problem 32

33. **(Mixing)** A tank contains 800 gal of water in which 200 lb of salt is dissolved. Two gallons of fresh water runs in per minute, and 2 gal of the mixture in the tank, kept uniform by stirring, runs out per minute. How much salt is left in the tank after 5 hours?

34. **WRITING PROJECT. Exponential Increase, Decay, Approach.** Collect, order, and present all the information on the ODE $y' = ky$ and its applications from the text and the problems. Add examples of your own.

35. **CAS EXPERIMENT. Graphing Solutions.** A CAS can usually graph solutions even if they are given by integrals that cannot be evaluated by the usual methods of calculus. Show this as follows.

(A) Graph the curves for the seven initial value problems $y' = e^{-x^2/2}$, $y(0) = 0, \pm 1, \pm 2, \pm 3$, common axes. Are these curves congruent? Why?

(B) Experiment with approximate curves of nth partial sums of the Maclaurin series obtained by termwise integration of that of y in (A); graph them and describe qualitatively the accuracy for a fixed interval $0 \leqq x \leqq b$ and increasing n, and then for fixed n and increasing b.

(C) Experiment with $y' = \cos(x^2)$ as in (B).

(D) Find an initial value problem with solution $y = e^{x^2}\int_0^x e^{-t^2}\,dt$ and experiment with it as in (B).

36. **TEAM PROJECT. Torricelli's Law.** Suppose that the tank in Example 5 is hemispherical, of radius R, initially full of water, and has an outlet of 5 cm^2 cross-sectional area at the bottom. (Make a sketch.) Set up the model for outflow. Indicate what portion of your work in Example 5 you can use (so that it can become part of the general method independent of the shape of the tank). Find the time t to empty the tank (a) for any R, (b) for $R = 1$ m. Plot t as function of R. Find the time when $h = R/2$ (a) for any R, (b) for $R = 1$ m.

1.4 Exact ODEs. Integrating Factors

We remember from calculus that if a function $u(x, y)$ has continuous partial derivatives, its **differential** (also called its *total differential*) is

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

From this it follows that if $u(x, y) = c = const$, then $du = 0$.

For example, if $u = x + x^2y^3 = c$, then

$$du = (1 + 2xy^3)\,dx + 3x^2y^2\,dy = 0$$

or

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2y^2},$$

an ODE that we can solve by going backward. This idea leads to a powerful solution method as follows.

A first-order ODE $M(x, y) + N(x, y)y' = 0$, written as (use $dy = y'\,dx$ as in Sec. 1.3)

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad (1)$$

Differentiate this with respect to y and use (4b) to get

$$\frac{\partial u}{\partial y} = x + \frac{dk}{dy} = N = x - e^{-y}, \qquad \frac{dk}{dy} = -e^{-y}, \qquad k = e^{-y} + c^*.$$

Hence the general solution is

$$u(x, y) = e^x + xy + e^{-y} = c.$$

Step 3. Particular solution. The initial condition $y(0) = 1$ gives $u(0, -1) = 1 + 0 + e = 3.72$. Hence the answer is $e^x + xy + e^{-y} = 1 + e = 3.72$. Figure 15 shows several particular solutions obtained as level curves of $u(x, y) = c$, obtained by a CAS, a convenient way in cases in which it is impossible or difficult to cast a solution into explicit form. Note the curve that (nearly) satisfies the initial condition.

Step 4. Checking. Check by substitution that the answer satisfies the given equation as well as the initial condition. ■

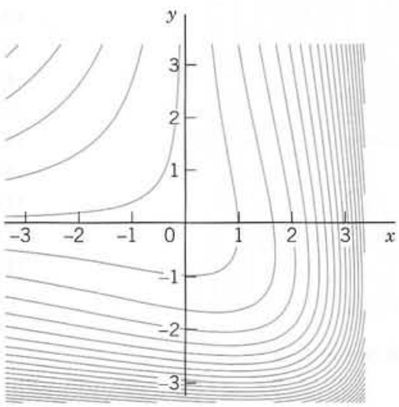

Fig. 15. Particular solutions in Example 5

PROBLEM SET 1.4

1–20 EXACT ODEs. INTEGRATING FACTORS

Test for exactness. If exact, solve. If not, use an integrating factor as given or find it by inspection or from the theorems in the text. Also, if an initial condition is given, determine the corresponding particular solution.

1. $x^3\,dx + y^3\,dy = 0$
2. $(x - y)(dx - dy) = 0$
3. $-\pi \sin \pi x \sinh y\,dx + \cos \pi x \cosh y\,dy = 0$
4. $(e^y - ye^x)\,dx + (xe^y - e^x)\,dy = 0$
5. $9x\,dx + 4y\,dy = 0$
6. $e^x(\cos y\,dx - \sin y\,dy) = 0$
7. $e^{-2\theta}\,dr - 2re^{-2\theta}\,d\theta = 0$
8. $(2x + 1/y - y/x^2)\,dx + (2y + 1/x - x/y^2)\,dy = 0$
9. $(-y/x^2 + 2\cos 2x)\,dx + (1/x - 2\sin 2y)\,dy = 0$
10. $-2xy \sin(x^2)\,dx + \cos(x^2)\,dy = 0$
11. $-y\,dx + x\,dy = 0$
12. $(e^{x+y} - y)\,dx + (xe^{x+y} + 1)\,dy = 0$
13. $-3y\,dx + 2x\,dy = 0, \quad F(x, y) = y/x^4$
14. $(x^4 + y^2)\,dx - xy\,dy = 0, \quad y(2) = 1$
15. $e^{2x}(2\cos y\,dx - \sin y\,dy) = 0, \quad y(0) = 0$
16. $-\sin xy\,(y\,dx + x\,dy) = 0, \quad y(1) = \pi$
17. $(\cos \omega x + \omega \sin \omega x)\,dx + e^x\,dy = 0, \; y(0) = 1$
18. $(\cos xy + x/y)\,dx + (1 + (x/y)\cos xy)\,dy = 0$
19. $e^{-y}\,dx + e^{-x}(-e^{-y} + 1)\,dy = 0, \quad F = e^{x+y}$
20. $(\sin y \cos y + x\cos^2 y)\,dx + x\,dy = 0$
21. Under what conditions for the constants A, B, C, D is $(Ax + By)\,dx + (Cx + Dy)\,dy = 0$ exact? Solve the exact equation.

22. CAS PROJECT. Graphing Particular Solutions Graph particular solutions of the following ODE, proceeding as explained.

$$y \cos x \, dx + \frac{1}{y} \, dy = 0 \qquad (21)$$

(a) Test for exactness. If necessary, find an integrating factor. Find the general solution $u(x, y) = c$.

(b) Solve (21) by separating variables. Is this simpler than **(a)**?

(c) Graph contours $u(x, y) = c$ by your CAS. (Cf. Fig. 16.)

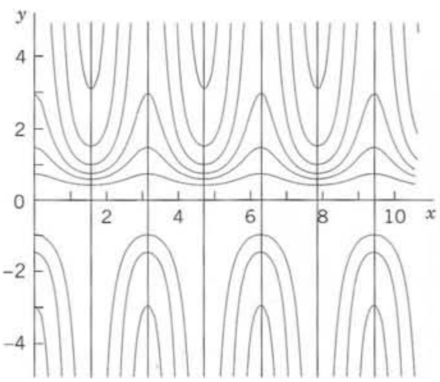

Fig. 16. Particular solutions in CAS Project 22

(d) In another graph show the solution curves satisfying $y(0) = \pm 1, \pm 2, \pm 3, \pm 4$. Compare the quality of **(c)** and **(d)** and comment.

(e) Do the same steps for another nonexact ODE of your choice.

23. WRITING PROJECT. Working Backward. Start from solutions $u(x, y) = c$ of your choice, find a corresponding exact ODE, destroy exactness by a multiplication or division. This should give you a feel for the form of ODEs you can reach by the method of integrating factors. (Working backward is useful in other areas, too; Euler and other great masters frequently did it.)

24. TEAM PROJECT. Solution by Several Methods. Show this as indicated. Compare the amount of work.

(A) $e^y(\sinh x \, dx + \cosh x \, dy) = 0$ as an exact ODE and by separation.

(B) $(1 + 2x) \cos y \, dx + dy/\cos y = 0$ by Theorem 2 and by separation.

(C) $(x^2 + y^2) \, dx - 2xy \, dy = 0$ by Theorem 1 or 2 and by separation with $v = y/x$.

(D) $3x^2 y \, dx + 4x^3 \, dy = 0$ by Theorems 1 and 2 and by separation.

(E) Search the text and the problems for further ODEs that can be solved by more than one of the methods discussed so far. Make a list of these ODEs. Find further cases of your own.

1.5 Linear ODEs. Bernoulli Equation. Population Dynamics

Linear ODEs or ODEs that can be transformed to linear form are models of various phenomena, for instance, in physics, biology, population dynamics, and ecology, as we shall see. A first-order ODE is said to be ***linear*** if it can be written

$$y' + p(x)y = r(x). \qquad \textbf{(1)}$$

The defining feature of this equation is that it is linear in both the unknown function y and its derivative $y' = dy/dx$, whereas p and r may be ***any*** given functions of x. If in an application the independent variable is time, we write t instead of x.

If the first term is $f(x)y'$ (instead of y'), divide the equation by $f(x)$ to get the **"standard form"** (1), with y' as the first term, which is practical.

For instance, $y' \cos x + y \sin x = x$ is a linear ODE, and its standard form is $y' + y \tan x = x \sec x$.

The function $r(x)$ on the right may be a force, and the solution $y(x)$ a displacement in a motion or an electrical current or some other physical quantity. In engineering, $r(x)$ is frequently called the **input,** and $y(x)$ is called the **output** or the *response* to the input (and, if given, to the initial condition).

EXAMPLE 5 Stable and Unstable Equilibrium Solutions. "Phase Line Plot"

The ODE $y' = (y - 1)(y - 2)$ has the stable equilibrium solution $y_1 = 1$ and the unstable $y_2 = 2$, as the direction field in Fig. 19 suggests. The values y_1 and y_2 are the zeros of the parabola $f(y) = (y - 1)(y - 2)$ in the figure. Now, since the ODE is autonomous, we can "condense" the direction field to a "phase line plot" giving y_1 and y_2, and the direction (upward or downward) of the arrows in the field, and thus giving information about the stability or instability of the equilibrium solutions. ■

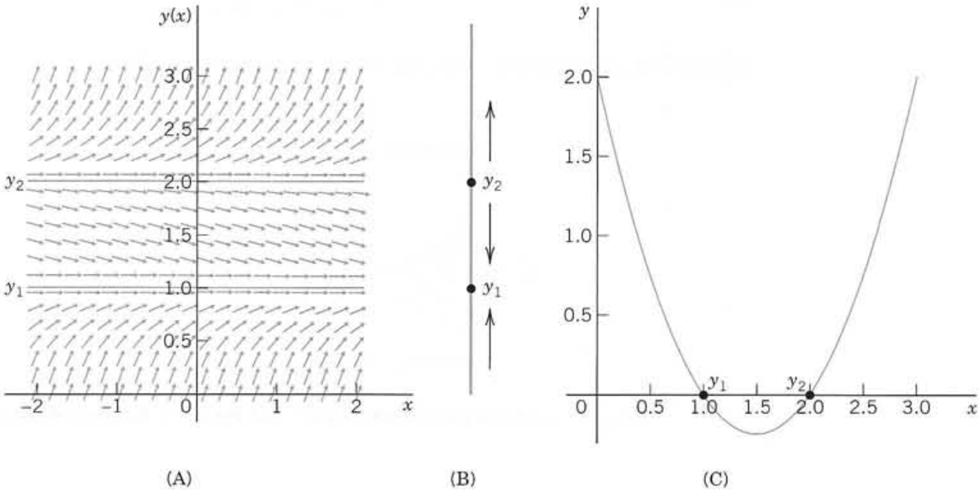

Fig. 19. Example 5. (A) Direction field. (B) "Phase line". (C) Parabola $f(y)$

A few further population models will be discussed in the problem set. For some more details of population dynamics, see C. W. Clark, *Mathematical Bioeconomics,* New York, Wiley, 1976.

Further important applications of linear ODEs follow in the next section.

PROBLEM SET 1.5

1. (CAUTION!) Show that $e^{-\ln x} = 1/x$ (not $-x$) and $e^{-\ln(\sec x)} = \cos x$.
2. **(Integration constant)** Give a reason why in (4) you may choose the constant of integration in $\int p\,dx$ to be zero.

3–17 GENERAL SOLUTION. INITIAL VALUE PROBLEMS

Find the general solution. If an initial condition is given, find also the corresponding particular solution and graph or sketch it. (Show the details of your work.)

3. $y' + 3.5y = 2.8$
4. $y' = 4y + x$
5. $y' + 1.25y = 5, \quad y(0) = 6.6$
6. $x^2y' + 3xy = 1/x, \quad y(1) = -1$
7. $y' + ky = e^{2kx}$
8. $y' + 2y = 4\cos 2x, \quad y(\frac{1}{4}\pi) = 2$
9. $y' = 6(y - 2.5)\tanh 1.5x$
10. $y' + 4x^2y = (4x^2 - x)e^{-x^2/2}$
11. $y' + 2y\sin 2x = 2e^{\cos 2x}, \quad y(0) = 0$
12. $y'\tan x = 2y - 8, \quad y(\frac{1}{2}\pi) = 0$
13. $y' + 4y\cot 2x = 6\cos 2x, \quad y(\frac{1}{4}\pi) = 2$
14. $y' + y\tan x = e^{-0.01x}\cos x, \quad y(0) = 0$
15. $y' + y/x^2 = 2xe^{1/x}, \quad y(1) = 13.86$
16. $y'\cos^2 x + 3y = 1, \quad y(\frac{1}{4}\pi) = \frac{4}{3}$
17. $x^3y' + 3x^2y = 5\sinh 10x$

18–24 NONLINEAR ODEs

Using a method of this section or separating variables, find the general solution. If an initial condition is given, find also the particular solution and sketch or graph it.

18. $y' + y = y^2, \quad y(0) = -1$

19. $y' = 5.7y - 6.5y^2$

20. $(x^2 + 1)y' = -\tan y, \quad y(0) = \frac{1}{2}\pi$

21. $y' + (x + 1)y = e^{x^2}y^3, \quad y(0) = 0.5$

22. $y' \sin 2y + x \cos 2y = 2x$

23. $2yy' + y^2 \sin x = \sin x, \quad y(0) = \sqrt{2}$

24. $y' + x^2y = (e^{-x^3} \sinh x)/(3y^2)$

25–36 FURTHER APPLICATIONS

25. (Investment programs) Bill opens a retirement savings account with an initial amount y_0 and then adds $\$k$ to the account at the beginning of every year until retirement at age 65. Assume that the interest is compounded continuously at the same rate R over the years. Set up a model for the balance in the account and find the general solution as well as the particular solution, letting $t = 0$ be the instant when the account is opened. How much money will Bill have in the account at age 65 if he starts at 25 and invests \$1000 initially as well as annually, and the interest rate R is 6%? How much should he invest initially and annually (same amounts) to obtain the same final balance as before if he starts at age 45? First, guess.

26. (Mixing problem) A tank (as in Fig. 9 in Sec. 1.3) contains 1000 gal of water in which 200 lb of salt is dissolved. 50 gal of brine, each gallon containing $(1 + \cos t)$ lb of dissolved salt, runs into the tank per minute. The mixture, kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time t (Fig. 20).

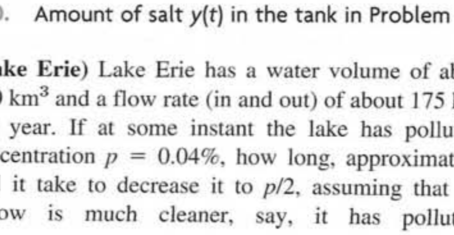

Fig. 20. Amount of salt $y(t)$ in the tank in Problem 26

27. (Lake Erie) Lake Erie has a water volume of about 450 km^3 and a flow rate (in and out) of about 175 km^3 per year. If at some instant the lake has pollution concentration $p = 0.04\%$, how long, approximately, will it take to decrease it to $p/2$, assuming that the inflow is much cleaner, say, it has pollution concentration $p/4$, and the mixture is uniform (an assumption that is only very imperfectly true)? First, guess.

28. (Heating and cooling of a building) Heating and cooling of a building can be modeled by the ODE

$$T' = k_1(T - T_a) + k_2(T - T_w) + P,$$

where $T = T(t)$ is the temperature in the building at time t, T_a the outside temperature, T_w the temperature wanted in the building, and P the rate of increase of T due to machines and people in the building, and k_1 and k_2 are (negative) constants. Solve this ODE, assuming $P = const$, $T_w = const$, and T_a varying sinusoidally over 24 hours, say, $T_a = A - C \cos(2\pi/24)t$. Discuss the effect of each term of the equation on the solution.

29. (Drug injection) Find and solve the model for drug injection into the bloodstream if, beginning at $t = 0$, a constant amount A g/min is injected and the drug is simultaneously removed at a rate proportional to the amount of the drug present at time t.

30. (Epidemics) A model for the spread of contagious diseases is obtained by assuming that the rate of spread is proportional to the number of contacts between infected and noninfected persons, who are assumed to move freely among each other. Set up the model. Find the equilibrium solutions and indicate their stability or instability. Solve the ODE. Find the limit of the proportion of infected persons as $t \to \infty$ and explain what it means.

31. (Extinction vs. unlimited growth) If in a population $y(t)$ the death rate is proportional to the population, and the birth rate is proportional to the chance encounters of meeting mates for reproduction, what will the model be? Without solving, find out what will eventually happen to a small initial population. To a large one. Then solve the model.

32. (Harvesting renewable resources. Fishing) Suppose that the population $y(t)$ of a certain kind of fish is given by the logistic equation (8), and fish are caught at a rate Hy proportional to y. Solve this so-called *Schaefer model*. Find the equilibrium solutions y_1 and y_2 (> 0) when $H < A$. The expression $Y = Hy_2$ is called the **equilibrium harvest** or **sustainable yield** *corresponding to H*. Why?

33. (Harvesting) In Prob. 32 find and graph the solution satisfying $y(0) = 2$ when (for simplicity) $A = B = 1$ and $H = 0.2$. What is the limit? What does it mean? What if there were no fishing?

34. (Intermittent harvesting) In Prob. 32 assume that you fish for 3 years, then fishing is banned for the next 3 years. Thereafter you start again. And so on. This is called *intermittent harvesting*. Describe qualitatively how the population will develop if intermitting is continued periodically. Find and graph the solution for the first 9 years, assuming that $A = B = 1$, $H = 0.2$, and $y(0) = 2$.

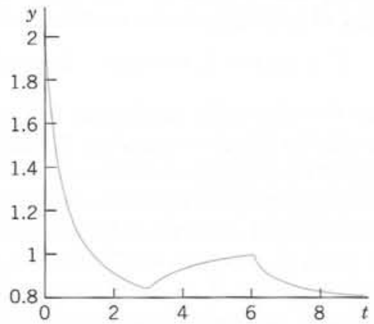

Fig. 21. Fish population in Problem 34

35. (Harvesting) If a population of mice (in multiples of 1000) follows the logistic law with $A = 1$ and $B = 0.25$, and if owls catch at a time rate of 10% of the population present, what is the model, its equilibrium harvest for that catch, and its solution?

36. (Harvesting) Do you save work in Prob. 34 if you first transform the ODE to a linear ODE? Do this transformation. Solve the resulting ODE. Does the resulting $y(t)$ agree with that in Prob. 34?

37–40 GENERAL PROPERTIES OF LINEAR ODEs

These properties are of practical and theoretical importance because they enable us to obtain new solutions from given ones. Thus in modeling, whenever possible, we prefer linear ODEs over nonlinear ones, which have no similar properties.

Show that nonhomogeneous linear ODEs (1) and homogeneous linear ODEs (2) have the following properties. Illustrate each property by a calculation for two or three equations of your choice. Give proofs.

37. The sum $y_1 + y_2$ of two solutions y_1 and y_2 of the homogeneous equation (2) is a solution of (2), and so is a scalar multiple ay_1 for any constant a. These properties are not true for (1)!

38. $y = 0$ (that is, $y(x) = 0$ for all x, also written $y(x) \equiv 0$) is a solution of (2) [not of (1) if $r(x) \neq 0$!], called the **trivial solution.**

39. The sum of a solution of (1) and a solution of (2) is a solution of (1).

40. The difference of two solutions of (1) is a solution of (2).

41. If y_1 is a solution of (1), what can you say about cy_1?

42. If y_1 and y_2 are solutions of $y_1' + py_1 = r_1$ and $y_2' + py_2 = r_2$, respectively (with the same p!), what can you say about the sum $y_1 + y_2$?

43. CAS EXPERIMENT. (a) Solve the ODE $y' - y/x = -x^{-1}\cos(1/x)$. Find an initial condition for which the arbitrary constant is zero. Graph the resulting particular solution, experimenting to obtain a good figure near $x = 0$.

(b) Generalizing (a) from $n = 1$ to arbitrary n, solve the ODE $y' - ny/x = -x^{n-2}\cos(1/x)$. Find an initial condition as in (a), and experiment with the graph.

44. TEAM PROJECT. Riccati Equation, Clairaut Equation. A **Riccati equation** is of the form

$$y' + p(x)y = g(x)y^2 + h(x). \tag{11}$$

A **Clairaut equation** is of the form

$$y = xy' + g(y'). \tag{12}$$

(a) Apply the transformation $y = Y + 1/u$ to the Riccati equation (11), where Y is a solution of (11), and obtain for u the linear ODE $u' + (2Yg - p)u = -g$. Explain the effect of the transformation by writing it as $y = Y + v$, $v = 1/u$.

(b) Show that $y = Y = x$ is a solution of $y' - (2x^3 + 1)y = -x^2y^2 - x^4 - x + 1$ and solve this Riccati equation, showing the details.

(c) Solve $y' + (3 - 2x^2 \sin x)y = -y^2 \sin x + 2x + 3x^2 - x^4 \sin x$, using (and verifying) that $y = x^2$ is a solution.

(d) By working "backward" from the u-equation find further Riccati equations that have relatively simple solutions.

(e) Solve the Clairaut equation $y = xy' + 1/y'$. *Hint.* Differentiate this ODE with respect to x.

(f) Solve the Clairaut equation $y'^2 - xy' + y = 0$ in Prob. 16 of Problem Set 1.1.

(g) Show that the Clairaut equation (12) has as solutions a family of straight lines $y = cx + g(c)$ and a singular solution determined by $g'(s) = -x$, where $s = y'$, that forms the envelope of that family.

45. (Variation of parameter) Another method of obtaining (4) results from the following idea. Write (3) as cy^*, where y^* is the exponential function, which is a solution of the homogeneous linear ODE $y^{*\prime} + py^* = 0$. Replace the arbitrary constant c in (3) with a function u to be determined so that the resulting function $y = uy^*$ is a solution of the nonhomogeneous linear ODE $y' + py = r$.

46. TEAM PROJECT. Transformations of ODEs. We have transformed ODEs to separable form, to exact form, and to linear form. The purpose of such transformations is an extension of solution methods to larger classes of ODEs. Describe the key idea of each of these transformations and give three typical examples of your choice for each transformation, showing each step (not just the transformed ODE).

EXAMPLE 2 Nonuniqueness

The initial value problem

$$y' = \sqrt{|y|}, \qquad y(0) = 0$$

has the two solutions

$$y = 0 \qquad \text{and} \qquad y^* = \begin{cases} x^2/4 & \text{if } x \geqq 0 \\ -x^2/4 & \text{if } x < 0 \end{cases}$$

although $f(x, y) = \sqrt{|y|}$ is continuous for all y. The Lipschitz condition (4) is violated in any region that includes the line $y = 0$, because for $y_1 = 0$ and positive y_2 we have

$$\text{(5)} \qquad \frac{|f(x, y_2) - f(x, y_1)|}{|y_2 - y_1|} = \frac{\sqrt{y_2}}{y_2} = \frac{1}{\sqrt{y_2}}, \qquad (\sqrt{y_2} > 0)$$

and this can be made as large as we please by choosing y_2 sufficiently small, whereas (4) requires that the quotient on the left side of (5) should not exceed a fixed constant M. ■

PROBLEM SET 1.7

1. **(Vertical strip)** If the assumptions of Theorems 1 and 2 are satisfied not merely in a rectangle but in a vertical infinite strip $|x - x_0| < a$, in what interval will the solution of (1) exist?
2. **(Existence?)** Does the initial value problem $(x - 1)y' = 2y$, $y(1) = 1$ have a solution? Does your result contradict our present theorems?
3. **(Common points)** Can two solution curves of the same ODE have a common point in a rectangle in which the assumptions of the present theorems are satisfied?
4. **(Change of initial condition)** What happens in Prob. 2 if you replace $y(1) = 1$ with $y(1) = k$?
5. **(Linear ODE)** If p and r in $y' + p(x)y = r(x)$ are continuous for all x in an interval $|x - x_0| \leqq a$, show that $f(x, y)$ in this ODE satisfies the conditions of our present theorems, so that a corresponding initial value problem has a unique solution. Do you actually need these theorems for this ODE?
6. **(Three possible cases)** Find all initial conditions such that $(x^2 - 4x)y' = (2x - 4)y$ has no solution, precisely one solution, and more than one solution.
7. **(Length of x-interval)** In most cases the solution of an initial value problem (1) exists in an x-interval larger than that guaranteed by the present theorems. Show this fact for $y' = 2y^2$, $y(1) = 1$ by finding the best possible α (choosing b optimally) and comparing the result with the actual solution.
8. **PROJECT. Lipschitz Condition. (A)** State the definition of a Lipschitz condition. Explain its relation to the existence of a partial derivative. Explain its significance in our present context. Illustrate your statements by examples of your own.

   (B) Show that for a ***linear*** ODE $y' + p(x)y = r(x)$ with continuous p and r in $|x - x_0| \leqq a$ a Lipschitz condition holds. This is remarkable because it means that for a *linear* ODE the continuity of $f(x, y)$ guarantees not only the existence but also the uniqueness of the solution of an initial value problem. (Of course, this also follows directly from (4) in Sec. 1.5.)

   (C) Discuss the uniqueness of solution for a few simple ODEs that you can solve by one of the methods considered, and find whether a Lipschitz condition is satisfied.
9. **(Maximum α)** What is the largest possible α in Example 1 in the text?
10. **CAS PROJECT. Picard Iteration. (A)** Show that by integrating the ODE in (1) and observing the initial condition you obtain

$$\text{(6)} \qquad y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

[7]RUDOLF LIPSCHITZ (1832–1903), German mathematician. Lipschitz and similar conditions are important in modern theories, for instance, in partial differential equations.

This form (6) of (1) suggests **Picard's iteration method**[8], which is defined by

$$(7)\quad y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt, \quad n = 1, 2, \cdots.$$

It gives approximations $y_1, y_2, y_3, \cdots$ of the unknown solution y of (1). Indeed, you obtain y_1 by substituting $y = y_0$ on the right and integrating—this is the first step—, then y_2 by substituting $y = y_1$ on the right and integrating—this is the second step—, and so on. Write a program of the iteration that gives a printout of the first approximations $y_0, y_1, \cdots, y_N$ as well as their graphs on common axes. Try your program on two initial value problems of your own choice.

(B) Apply the iteration to $y' = x + y$, $y(0) = 0$. Also solve the problem exactly.

(C) Apply the iteration to $y' = 2y^2$, $y(0) = 1$. Also solve the problem exactly.

(D) Find all solutions of $y' = 2\sqrt{y}$, $y(1) = 0$. Which of them does Picard's iteration approximate?

(E) Experiment with the conjecture that Picard's iteration converges to the solution of the problem for any initial choice of y in the integrand in (7) (leaving y_0 outside the integral as it is). Begin with a simple ODE and see what happens. When you are reasonably sure, take a slightly more complicated ODE and give it a try.

CHAPTER 1 REVIEW QUESTIONS AND PROBLEMS

1. Explain the terms *ordinary differential equation (ODE), partial differential equation (PDE), order, general solution,* and *particular solution.* Give examples. Why are these concepts of importance?

2. What is an initial condition? How is this condition used in an initial value problem?

3. What is a homogeneous linear ODE? A nonhomogeneous linear ODE? Why are these equations simpler than nonlinear ODEs?

4. What do you know about direction fields and their practical importance?

5. Give examples of mechanical problems that lead to ODEs.

6. Why do electric circuits lead to ODEs?

7. Make a list of the solution methods considered. Explain each method with a few short sentences and illustrate it by a typical example.

8. Can certain ODEs be solved by more than one method? Give three examples.

9. What are integrating factors? Explain the idea. Give examples.

10. Does every first-order ODE have a solution? A general solution? What do you know about uniqueness of solutions?

11–14 DIRECTION FIELDS

Graph a direction field (by a CAS or by hand) and sketch some of the solution curves. Solve the ODE exactly and compare.

11. $y' = 1 + 4y^2$ **12.** $y' = 3y - 2x$

13. $y' = 4y - y^2$ **14.** $y' = 16x/y$

15–26 GENERAL SOLUTION

Find the general solution. Indicate which method in this chapter you are using. Show the details of your work.

15. $y' = x^2(1 + y^2)$

16. $y' = x(y - x^2 + 1)$

17. $yy' + xy^2 = x$

18. $-\pi \sin \pi x \cosh 3y\, dx + 3 \cos \pi x \sinh 3y\, dy = 0$

19. $y' + y \sin x = \sin x$ **20.** $y' - y = 1/y$

21. $3 \sin 2y\, dx + 2x \cos 2y\, dy = 0$

22. $xy' = x \tan(y/x) + y$

23. $(y \cos xy - 2x)\, dx + (x \cos xy + 2y)\, dy = 0$

24. $xy' = (y - 2x)^2 + y$ (Set $y - 2x = z$.)

25. $\sin(y - x)\, dx + [\cos(y - x) - \sin(y - x)]\, dy = 0$

26. $xy' = (y/x)^3 + y$

27–32 INITIAL VALUE PROBLEMS

Solve the following initial value problems. Indicate the method used. Show the details of your work.

27. $yy' + x = 0, \quad y(3) = 4$

28. $y' - 3y = -12y^2, \quad y(0) = 2$

29. $y' = 1 + y^2, \quad y(\frac{1}{4}\pi) = 0$

30. $y' + \pi y = 2b \cos \pi x, \quad y(0) = 0$

31. $(2xy^2 - \sin x)\, dx + (2 + 2x^2y)\, dy = 0$, $y(0) = 1$

32. $[2y + y^2/x + e^x(1 + 1/x)]\, dx + (x + 2y)\, dy = 0$, $y(1) = 1$

[8]EMILE PICARD (1856–1941), French mathematician, also known for his important contributions to complex analysis (see Sec. 16.2 for his famous theorem). Picard used his method to prove Theorems 1 and 2 as well as the convergence of the sequence (7) to the solution of (1). In precomputer times the iteration was of little *practical* value because of the integrations.