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Relativistic Dynamics:

In relativistic mechanics, the dynamical quantities are: mass, energy, momentum and force, from the point of view of special theory of relativity.

In Newtonian mechanics we have:

$$p = m v \quad , \quad K = \frac{1}{2} m v^2 \quad , \quad F = m a \quad , \quad F = \frac{dp}{dt}$$

Also m = constant that does not change for moving bodies.

What are the corresponding relations in the theory of relativity?

Newtonian Mechanics

* $\vec{p} = m \vec{v} = m \frac{d\vec{r}}{dt}$, $K = \frac{1}{2} m v^2 = \frac{p^2}{2m}$
* Vectors have $\underline{\underline{3}}$ components
* $\left.\begin{array}{l} \vec{r} = x \hat{i} + y \hat{j} + z \hat{k} \\ \vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \end{array}\right\}$ Cartesian coordinates (x, y, z)
* mass: Absolute (quantity of matter).
  Independent of velocity.
* Time: Absolute in different inertial frames

Einstein's Mechanics

* Vectors have $\underline{\underline{4}}$ components (x, y, z, t).
* mass: Not absolute (depends on velocity).
* Time: Not absolute (depends on velocity).

Example: $x' = \gamma(x - vt)$

$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{xv}{c^2}\right).$$

Now we know that $m = \dfrac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m_0$.

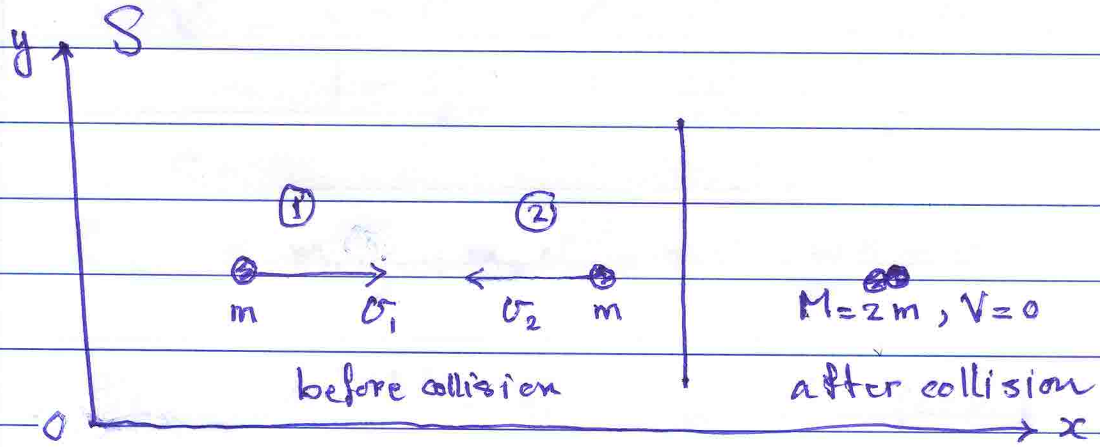
m_0 = rest mass

m = mass of the object moving with velocity v.

Prove the above relation through an example

Consider a completely inelastic collision between two masses (equal) moving along x-direction ith velocities v and $-v$.

In the stationary frame S

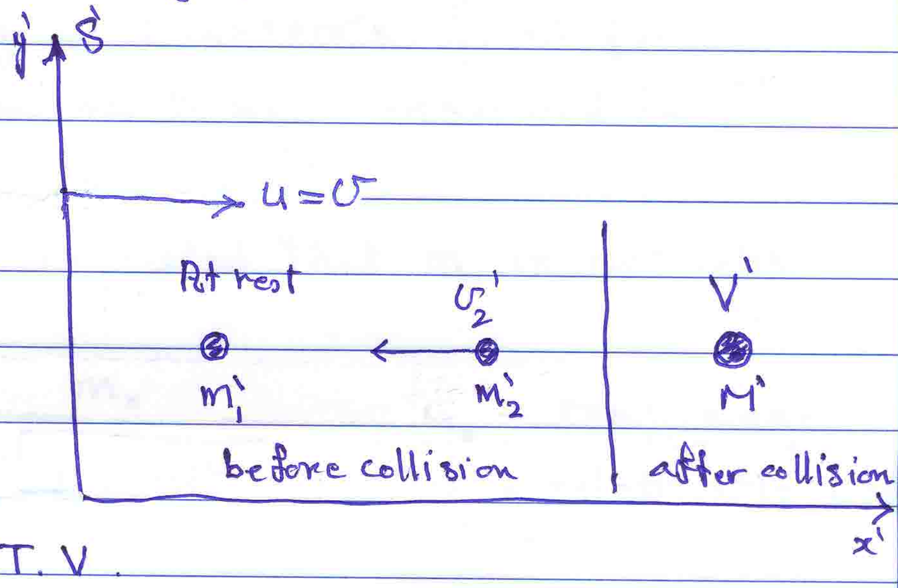


$|v_1| = |v_2| = v$ $\qquad$ $v_2 = -v_1 = -v$

Linear momentum is conserved

$$m v_1 + m v_2 = 2m(0).$$

In the moving frame S'



Using L.T.V.

$$v_1' = \frac{v_1 - v}{1 - \frac{v_1 v}{c^2}} = \frac{v - v}{1 - \frac{v^2}{c^2}} = 0 \qquad (v_1 = v)$$

$$v_2' = \frac{v_2 - v}{1 - \frac{v_2 v}{c^2}} = \frac{-v - v}{1 + \frac{v^2}{c^2}} \qquad (v_2 = -v)$$

$$= \frac{-2v}{1 + v^2/c^2}$$

$$V' = \frac{V - v}{1 - \frac{V v}{c^2}} = \frac{0 - v}{1 - \frac{(0) v}{c^2}} = -v \qquad (V = 0)$$

$\Rightarrow$ In S:

$$P_{before} = m_1 v_1 + m_2 v_2 = m v - m v = 0$$

$$P_{after} = (2m) V = 0 \qquad \checkmark \qquad (\text{Ok})$$

In S':

$$P'_{before} = m_1 v_1' + m_2 v_2' = m(0) + m\left(\frac{-2v}{1 + \frac{v^2}{c^2}}\right)$$

$$P'_{after} = 2mV' = -2mv$$

$$\Rightarrow P'_{before} \neq P'_{after} \; !!!$$

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According to Einstein's first postulate, linear momentum must be conserved in S'

Einstein suggested that m is not absolute $\Rightarrow$

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \qquad m_0 = \text{rest mass}, \quad m = \text{relativistic mass}$$

If this suggestion leads to conservation of linear momentum in S', then it is acceptable.

We have, in our problem

$$m_1 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}, \quad m_2 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$v_1 = v_2 = v$$

$$\boxed{M = m_1 + m_2 = \frac{2m_0}{\sqrt{1-\frac{v^2}{c^2}}}}$$

Since M is at rest in S $\Rightarrow$ $M = M_0$ —— **

In S', m_1' is at rest $\Rightarrow$ $m_1' = m_0$

m_2' is moving with $v_2' = \dfrac{-2v}{\left(1+\frac{v^2}{c^2}\right)}$

$\Rightarrow$

$$m_2' = \frac{m_0}{\sqrt{1-\frac{1}{c^2}\left(\frac{-2v}{1+\frac{v^2}{c^2}}\right)^2}}$$

$$= \frac{m_0}{\sqrt{1 - \frac{4v^2}{c^2\left(1+\frac{v^2}{c^2}\right)^2}}}$$

$$= \frac{m_0}{\sqrt{1 - \frac{4v^2c^2}{(c^2+v^2)}}}$$

$$= \frac{m_0}{\sqrt{\frac{(c^2+v^2)^2 - 4v^2c^2}{(c^2+v^2)^2}}}$$

$$= \frac{m_0}{\sqrt{\left(\frac{c^2-v^2}{c^2+v^2}\right)^2}}$$

$$= m_0\left(\frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\right)$$

The combined mass M' is moving with $V' = -v$, so

$$M' = \frac{M_0}{\sqrt{1-\frac{V'^2}{c^2}}} = \frac{M_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Substitute for $M_0 = \frac{2m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\Rightarrow M' = \frac{2m_0}{\left(1-\frac{v^2}{c^2}\right)}$$

Now

$$P'_{before} = m'_1 v'_1 + m'_2 v'_2$$

$$= m_o(0) + m_o\left(\frac{1+v^2/c^2}{1-v^2/c^2}\right)\left(\frac{-2v}{1+v^2/c^2}\right)$$

$$= \frac{-2m_o v}{(1-v^2/c^2)}$$

$$P'_{after} = M'V' = \frac{2m_o}{\left(1-\frac{v^2}{c^2}\right)}(-v)$$

$$= \frac{-2m_o v}{\left(1-\frac{v^2}{c^2}\right)}$$

$\Rightarrow$

$$P'_{before} = P'_{after} \quad \checkmark$$

$\Rightarrow$

Momentum is conserved in S'

Summary:

$$m = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{relativistic mass}$$

$$P = mv = \frac{m_o v}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{relativistic momentum}$$

Calculate now ΔK

Assuming that the object starts from rest

$\Rightarrow$

$$\Delta K = K_f - K_i = K - 0 = K.$$

From work-energy theorem

$$\Delta K = K = W = \int F\,dx = \int \frac{dP}{dt}\,dx$$

$\Rightarrow$

$$K = \int \frac{dP}{dt}\,dx = \int v\,dp.$$

Integrating by parts $d(pv) = \underset{\sim}{v\,dp} + p\,dv$

$\Rightarrow$

$$K = \int d(pv) - \int_{v_i=0}^{v_f=v} p\,dv$$

$$= pv - \int_0^v p\,dv$$

$$= \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\,v - \int_0^v \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\,dv$$

$$= \frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 \underbrace{\int_0^v \frac{v\,dv}{\sqrt{1-\frac{v^2}{c^2}}}}$$

Change variables $v = c\cos\theta$

$\Rightarrow dv = -c\sin\theta\,d\theta$

$\Rightarrow$

$$-m_0\int_0^v \frac{v\,dv}{\sqrt{1-\frac{v^2}{c^2}}} = +m_0 c^2 \int \frac{\sin\theta\cos\theta}{\sqrt{1-\cos^2\theta}}\,d\theta$$

$$-m_o \int \frac{v\,dv}{\sqrt{1-\frac{v^2}{c^2}}} = +m_o c^2 \int \cos\theta \, d\theta$$

$$= +m_o c^2 \left[\sin\theta\right]$$

$$= +m_o c^2 \left[\sqrt{1-\cos^2\theta}\right]$$

$$= +m_o c^2 \left[\sqrt{1-\frac{v^2}{c^2}}\right]_o^v$$

$$= m_o c^2 \sqrt{1-\frac{v^2}{c^2}} - m_o c^2$$

$$\Rightarrow \quad K = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_o c^2 \sqrt{1-\frac{v^2}{c^2}} - m_o c^2$$

$$= \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} \left[\frac{v^2}{c^2} + \left(1-\frac{v^2}{c^2}\right)\right] - m_o c^2$$

$$\boxed{K = mc^2 - m_o c^2}$$ relativistic Kinetic energy.

$$\Rightarrow \quad K = mc^2 - E_o \quad , \quad E_o = \text{rest mass energy.}$$

or

$$\boxed{E = K + E_o = mc^2} \quad , \quad E = \text{total energy.}$$

$$\boxed{E = mc^2} \qquad m = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}$$

Go back to the completely inelastic collision problem and see if the total energy is conserved in the collision.

In S:

$$E_{m_1} = m_1 c^2 = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$E_{m_2} = m_2 c^2 = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$E_M = M_o c^2 = \frac{2 m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$\Rightarrow$

$$E_{m_1} + E_{m_2} = E_M .$$

$\Rightarrow$ Total energy is conserved in S.

In S':

$$E_{m_1'} = m_o c^2$$

$$E_{m_2'} = m_o c^2 \left(\frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} \right)$$

$$E_{M'} = \frac{2 m_o c^2}{(1 - v^2/c^2)}$$

$$\Rightarrow E_{m_1'} + E_{m_2'} = m_0c^2 + m_0c^2\left(\frac{1+v^2/c^2}{1-v^2/c^2}\right)$$

$$= m_0c^2\left[1 + \frac{(1+\frac{v^2}{c^2})}{(1-\frac{v^2}{c^2})}\right]$$

$$= m_0c^2\left[\frac{1-\frac{v^2}{c^2}+1+v^2/c^2}{1-\frac{v^2}{c^2}}\right]$$

$$= \frac{2m_0c^2}{(1-\frac{v^2}{c^2})}$$

$\Rightarrow$

E_{total} is conserved in S'

We can also derive an important relation between relativistic total energy and relativistic momentum which is

$$\boxed{E^2 = p^2c^2 + (m_0c^2)^2}$$

Proof:

$$p = \frac{m_0 v}{\sqrt{1-\frac{v^3}{c^2}}}, \quad m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$E = mc^2$$

$\Rightarrow$

$$p^2c^2 + (m_0c^2)^2 = \frac{m_0^2v^2c^2}{(1-\frac{v^2}{c^2})} + m_0^2c^4$$

$$= \frac{m_0v^2c^2 + m_0^2c^4(1-\frac{v^2}{c^2})}{(1-\frac{v^2}{c^2})}$$

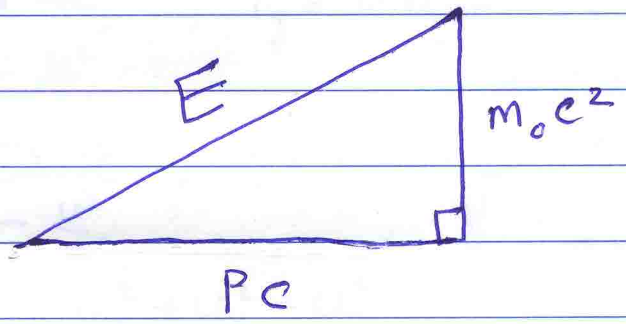
$$p^2c^2 + (m_0c^2)^2 = \frac{m_0^2\cancel{v^2c^2} + m_0^2c^4 - \cancel{m_0^2v^2c^2}}{\left(1-\frac{v^2}{c^2}\right)}$$

$$= \left(\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2$$

$$= (mc^2)^2$$

$$= E^2$$

$$\Rightarrow \boxed{E^2 = p^2c^2 + (m_0c^2)^2}$$



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Next we consider the following

(P_x, P_y, P_z, E) as the components of momentum and total energy of a particle in S frame

This is called (momentum-Energy 4-vector)
Remember that we learned about space-time 4-vector (x, y, z, t)

Now in S', what is the momentum-energy 4-vector?

Solution: S' → v along +ve x-axis.

In S:

$$P_x = m u_x = \frac{m_0 u_x}{\sqrt{1-\frac{u^2}{c^2}}} , \quad P_y = m u_y = \frac{m_0 u_y}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$P_z = m u_z = \frac{m_0 u_z}{\sqrt{1-\frac{u^2}{c^2}}} , \quad E = mc^2 = \frac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}}$$

Of course $u^2 = u_x^2 + u_y^2 + u_z^2$

In S':

$$P'_x = m' u'_x = \frac{m_0 u'_x}{\sqrt{1-\frac{u'^2}{c^2}}} , \quad P'_y = m' u'_y = \frac{m_0 u'_y}{\sqrt{1-\frac{u'^2}{c^2}}}$$

$$P'_z = m' u'_z = \frac{m_0 u'_z}{\sqrt{1-\frac{u'^2}{c^2}}} , \quad E' = m'c^2 = \frac{m_0 c^2}{\sqrt{1-\frac{u'^2}{c^2}}}$$

Of course $u'^2 = u_x'^2 + u_y'^2 + u_z'^2$

$\Rightarrow$

$$u'^2 = \frac{(u_x - v)^2}{\left(1 - \frac{u_x v}{c^2}\right)^2} + \frac{\left(u_y\sqrt{1 - v^2/c^2}\right)^2}{\left(1 - \frac{u_x v}{c^2}\right)^2} + \frac{\left(u_z^2\sqrt{1 - v^2/c^2}\right)^2}{\left(1 - \frac{u_x v}{c^2}\right)^2}$$

$$= \frac{(u_x - v)^2 + u_y^2\left(1 - \frac{v^2}{c^2}\right) + u_z^2\left(1 - \frac{v^3}{c^2}\right)}{\left(1 - \frac{u_x v}{c^2}\right)^2}$$

$$\left(1 - \frac{u'^2}{c^2}\right) = 1 - \frac{\left(\frac{u_x}{c} - \frac{v}{c}\right)^2 + \left(\frac{u^2}{c^2} - \frac{u_x^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{u_x v}{c^2}\right)^2}$$

$$= \frac{\left(1 - \frac{u_x v}{c^2}\right)^2 - \left(\frac{u_x}{c} - \frac{v}{c}\right)^2 - \left(\frac{u^2}{c^2} - \frac{u_x^2}{c^2}\right)\left(1 - \frac{v^3}{c^2}\right)}{\left(1 - \frac{u_x v}{c^2}\right)^2}$$

$$= \frac{1 - \cancel{2\frac{u_x v}{c^2}} + \cancel{\frac{u_x^2 v^3}{c^4}} - \cancel{\frac{u_x^2}{c^2}} - \frac{v^2}{c^2} + \cancel{2\frac{u_x v}{c^2}}}{\left(1 - \frac{u_x v}{c^2}\right)^2}$$

$$\frac{- \frac{u^2}{c^2} + \cancel{\frac{u_x^2}{c^2}} + \frac{u^2 v^2}{c^4} - \cancel{\frac{u_x^2 v^2}{c^4}}}{}$$

$$= \frac{\left(1 - \frac{v^2}{c^2}\right) - \frac{u^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{u_x v}{c^2}\right)^2}$$

$$= \frac{\left(1 - \frac{v^3}{c^2}\right)\left(1 - \frac{u^2}{c^2}\right)}{\left(1 - \frac{u_x v}{c^2}\right)^2}$$

$$\Rightarrow \sqrt{1 - \frac{u'^2}{c^2}} = \sqrt{\frac{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u^2}{c^2}\right)}{\left(1 - \frac{u_x v}{c^2}\right)^2}}$$

$$= \frac{\sqrt{1 - v^2/c^2}\,\sqrt{1 - u^2/c^2}}{\left(1 - \frac{u_x v}{c^2}\right)}$$

$$\sqrt{1-\frac{u'^2}{c^2}} = \frac{\sqrt{1-u^2/c^2}}{\gamma\left(1-\frac{u_x v}{c^2}\right)}$$

Similarly we can show that

$$\sqrt{1-\frac{u^2}{c^2}} = \frac{\sqrt{1-u'^2/c^2}}{\gamma\left(1+\frac{u'_x v}{c^2}\right)}$$

Go back to our problem (momentum-energy 4-vector)

<u>In S':</u>

$$p'_x = \frac{m_0 u'_x}{\sqrt{1-\frac{u'^2}{c^2}}} \quad , \quad u'_x = \frac{u_x - v}{\sqrt{1-\frac{u_x v}{c^2}}}$$

and

$$\frac{1}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{\left(1-\frac{u_x v}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}}$$

$\Rightarrow$

$$p'_x = m_0 \frac{(u_x - v)}{\cancel{\left(1-\frac{u_x v}{c^2}\right)}} \cdot \frac{\cancel{\left(1-\frac{u_x v}{c^2}\right)}}{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}}$$

$$= \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \cdot \frac{(u_x - v)}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \gamma\,(m u_x - m v)$$

$$= \gamma\left(p_x - \frac{E}{c^2} v\right).$$

Similarly, we can show that

$$P_y' = P_y \quad , \quad P_z' = P_z$$

Also

$$E' = m'c^2 = \frac{m_o c^2}{\sqrt{1 - \frac{u'^2}{c^2}}}$$

$$= \frac{m_o c^2}{\sqrt{1-\frac{u^2}{c^2}}} \frac{\left(1 - \frac{u_x v}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \gamma m c^2 \left(1 - \frac{u_x v}{c^2}\right)$$

$$\boxed{E' = \gamma\left(E - P_x v\right)}$$

The inverse relation is

$$E = \gamma\left(E' + P_x' v\right)$$

Note that (P_x , P_y , P_z) transform like (x, y, z) and E transforms like t.

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Also

$$p'^2 - \frac{E'^2}{c^2} = p_x'^2 + p_y'^2 + p_z'^2 - \frac{E'^2}{c^2}$$

$$= \gamma^2\left(p_x - \frac{Ev}{c^2}\right)^2 + p_y^2 + p_z^2 - \frac{\gamma^2}{c^2}\left(E - p_x v\right)^2$$

$$= \gamma^2\left(p_x^2 - \frac{2p_xEv}{c^2} + \frac{E^2v^2}{c^4}\right) + p_y^2 + p_z^2$$

$$- \gamma^2\left(\frac{E^2}{c^2} - 2\frac{p_xEv}{c^2} + \frac{p_x^2v^2}{c^2}\right)$$

$$= \gamma^2 p_x^2 - \gamma^2\left(\cancel{\frac{2p_xEv}{c^2}}\right) + \gamma^2\left(\frac{E^2v^2}{c^4}\right) + p_y^2 + p_z^2$$

$$- \gamma^2\frac{E^2}{c^2} + \gamma^2\left(\cancel{\frac{2p_xEv}{c^2}}\right) - \gamma^2\frac{p_x^2v^2}{c^2}$$

$$= \gamma^2 p_x^2\left(1 - \frac{v^2}{c^2}\right) - \gamma^2\frac{E^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)$$

$$+ p_y^2 + p_z^2$$

But $\gamma^2\left(1 - \frac{v^2}{c^2}\right) = 1$.

$\Rightarrow$

$$p'^2 - \frac{E'^2}{c^2} = p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2}$$

or

$$\boxed{p'^2 - \frac{E'^2}{c^2} = p^2 - \frac{E^2}{c^2}}$$

Transformation of forces in different inertial frames

In frame S:

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

In frame S':

$$\vec{F}' = \frac{d\vec{P}'}{dt'} = \frac{\left(\frac{dP'}{dt}\right)}{\left(\frac{dt'}{dt}\right)}$$

$$F'_x = \frac{\left(\frac{dP'_x}{dt}\right)}{\left(\frac{dt'}{dt}\right)}$$

$$P'_x = \gamma\left(P_x - \frac{Ev}{c^2}\right)$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\Rightarrow \quad \frac{dP'_x}{dt} = \gamma\left(\frac{dP_x}{dt} - \frac{v}{c^2}\frac{dE}{dt}\right)$$

$$\frac{dt'}{dt} = \gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)$$

Substitute $\Rightarrow$

$$F'_x = \frac{\gamma\left(\frac{dP_x}{dt} - \frac{v}{c^2}\frac{dE}{dt}\right)}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}$$

$$F'_x = \frac{F_x - \frac{v}{c^2}\frac{dE}{dt}}{\left(1 - \frac{v}{c^2}u_x\right)} \quad \text{————} (1)$$

Since $E^2 = P^2c^2 + m_0^2c^4 = c^2(\vec{P}\cdot\vec{P}) + m_0^2c^4$

$$\Rightarrow \quad 2E\frac{dE}{dt} = c^2\vec{P}\cdot\frac{d\vec{P}}{dt} + c^2\frac{d\vec{P}}{dt}\cdot\vec{P}$$

$$\cancel{2}E\frac{dE}{dt} = \cancel{2}c^2\vec{P}\cdot\frac{d\vec{P}}{dt} \quad , \quad E = mc^2$$

$$\frac{dE}{dt} = \frac{\vec{P}}{m}\cdot\frac{d\vec{P}}{dt} = \vec{F}\cdot\vec{u}$$

Substitute in (1) $\Rightarrow$

$$F_x' = \frac{F_x - \frac{v}{c^2}(\vec{F}\cdot\vec{u})}{\left(1 - \frac{v}{c^2}u_x\right)}$$

Similarely $\Rightarrow$

$$F_y' = \frac{F_y}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}$$

$$F_z' = \frac{F_z}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}$$

The inverse relations are: $v \to -v$

$$F_x = \frac{F_x' + \frac{v}{c^2}(\vec{F}'\cdot\vec{u}')}{\left(1 + \frac{v}{c^2}u_x'\right)}$$

$$F_y = \frac{F_y'}{\gamma\left(1 + \frac{v}{c^2}u_x'\right)}$$

$$F_z = \frac{F_z'}{\gamma\left(1 + \frac{v}{c^2}u_x'\right)}$$

If the particle in frame S is at rest $\Rightarrow \vec{u} = 0$

$$\Rightarrow \quad F_x' = F_x$$

$$F_y' = \frac{F_y}{\gamma}$$

$$F_z' = \frac{F_z}{\gamma}$$

Clarify two basic concepts in Physics

* Concept of invariance
* Concept of conservation.

* A quantity is invariant if it is the same in different inertial frames.
  Example: Rest mass of a particle.

* A quantity is conserved if it remains unchanged in time, i.e. it remains the same after some interaction process.
  Example: Linear momentum, Total energy.

* Invariant quantities, such as rest mass, are not necessarily conserved.

* Conserved quantities, such as total energy and momentum, are not invariant since they change from one inertial frame to another.

Example: The quantity $E^2 - p^2c^2$

Since $E^2 = p^2c^2 + E_0^2$ $\qquad E_0 = m_0 c^2$

$\Rightarrow E^2 - p^2c^2 = E_0^2 = E'^2 - p'^2c^2$

i.e. the quantity $E^2 - p^2c^2$ is an invariant quantity.

Photons:

* Light is a wave that carries energy and exerts a pressure when it is incident on a surface or object.

* Photons are considered as massless particles

* From $E^2 = p^2c^2 + m_0^2c^4 \Rightarrow$

For photons

$$E = pc \qquad \text{or} \qquad p = \frac{E}{c}$$

* Energy carried by photons $E = h\nu$, where h is Planck's constant, ν = frequency.

$\Rightarrow$

$$p = \frac{h\nu}{c}$$

But $c = \nu\lambda \Rightarrow \frac{c}{\nu} = \lambda$.

$\Rightarrow$

$$\boxed{p = \frac{h}{\lambda}}$$