Motion in Two Dimensions

In one dimension $\bar{v} = \frac{\Delta x}{\Delta t}$

In two dimensions $\bar{\vec{v}} = \frac{\Delta \vec{r}}{\Delta t}$

where

$$\Delta \vec{r} = \Delta x\,\hat{i} + \Delta y\,\hat{j} \qquad (\vec{r} = x\,\hat{i} + y\,\hat{j})$$

$$\Rightarrow \quad \bar{\vec{v}} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} = \bar{v}_x\,\hat{i} + \bar{v}_y\,\hat{j}.$$

where

$$\bar{v}_x = \frac{\Delta x}{\Delta t}, \quad \bar{v}_y = \frac{\Delta y}{\Delta t}$$

Similarly for instantaneous velocity $\Rightarrow$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = v_x\,\hat{i} + v_y\,\hat{j}.$$

Note:

$$\text{Speed} = v = |\vec{v}| = \left|\frac{d\vec{r}}{dt}\right| = \text{magnitude of } \vec{v}.$$

Also from before

$$v = \sqrt{v_x^2 + v_y^2}, \quad \tan\theta_x = \frac{v_y}{v_x}$$

For the average acceleration in two dimensions

$$\bar{\vec{a}} = \frac{\Delta \vec{v}}{\Delta t}$$

For the instantaneous acceleration $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$

In terms of components

$$\bar{a}_x = \frac{\Delta v_x}{\Delta t} \quad , \quad \bar{a}_y = \frac{\Delta v_y}{\Delta t}$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \quad , \quad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$$

Motion in Two Dimensions: (Summary)

Parameter	1-Dimension	2-Dimensions
Position	x	$\vec{r} = x\hat{i} + y\hat{j}$
Average velocity	$\bar{v} = \frac{\Delta x}{\Delta t}$	$\bar{\vec{v}} = \frac{\Delta \vec{r}}{\Delta t}$
Displacement	$\Delta x = x_f - x_i$	$\Delta \vec{r} = \Delta x\hat{i} + \Delta y\hat{j}$
Instantaneous velocity	$v = \frac{dx}{dt}$	$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = \frac{d\vec{r}}{dt} = v_x\hat{i} + v_y\hat{j}$
Average Acceleration	$\bar{a} = \frac{\Delta v}{\Delta t}$	$\bar{\vec{a}} = \frac{\Delta v_x}{\Delta t}\hat{i} + \frac{\Delta v_y}{\Delta t}\hat{j}$
Instantaneous Acceleration	$a = \frac{dv}{dt}$	$\vec{a} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} = \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} = a_x\hat{i} + a_y\hat{j}$

Motion with constant acceleration in two dimensions

Since $\vec{a}$ is a vector, then $\vec{a}$ = constant means that a_x = constant

a_y = constant

a_z = constant

$\Rightarrow$ We can treat each dimension alone and write down the equations of motion

For a two dimensional motion

$\Rightarrow$

$$\vec{v}_f = \vec{v}_i + \vec{a}\,t$$

$$\vec{r}_f = \vec{r}_i + \vec{v}_i t + \frac{1}{2}\vec{a}\,t^2$$

$$\vec{v}_f^{\,2} = \vec{v}_i^{\,2} + 2\,\vec{a}\cdot(\vec{r}_f - \vec{r}_i).$$

In terms of components

For x-component	For y-component
$v_{x_f} = v_{x_i} + a_x t$	$v_{yf} = v_{yi} + a_y t$
$x_f = x_i + v_{xi} t + \frac{1}{2} a_x t^2$	$y_f = y_i + v_{yi} t + \frac{1}{2} a_y t^2$
$v_{xf}^2 = v_{xi}^2 + 2 a_x (x_f - x_i)$	$v_{yf}^2 = v_{yi}^2 + 2 a_y (y_f - y_i)$.

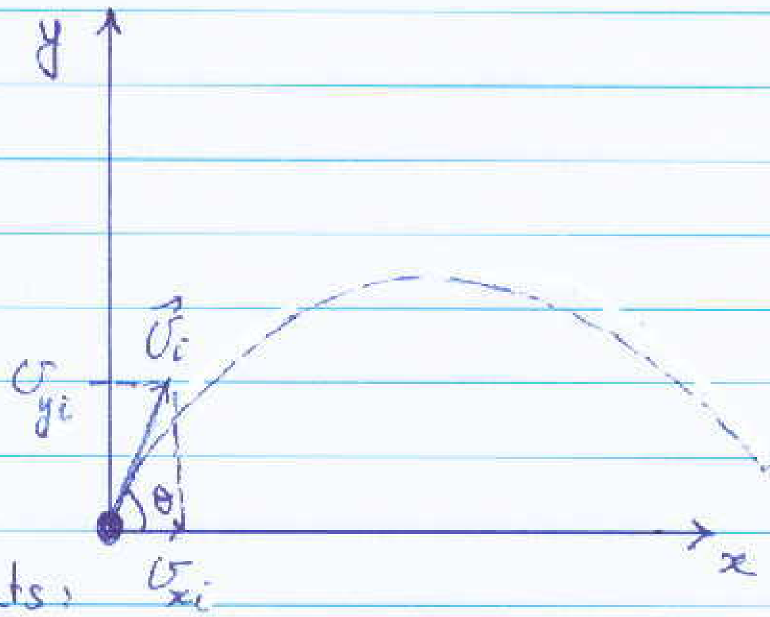
Applications on Motion in Two Dimensions

1) Projectile motion (magnitude of $\vec{v}$ changes)
2) Circular motion (direction of $\vec{v}$ changes)

1) Projectile motion

(In x-y plane)
(i.e. set $v_z = 0$, $z = 0$)

$$\Rightarrow a_x = 0$$
$$a_y = -g$$



In terms of components:

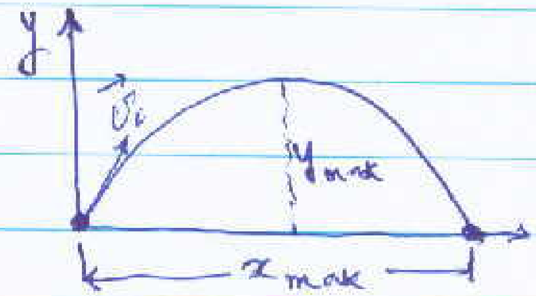
x-motion	y-motion
$v_{xf} = v_{xi} + (0)t = v_{xi}$	$v_{yf} = v_{yi} - gt$
$x_f = x_i + v_{xi}t + \frac{1}{2}(0)t^2$ $= x_i + v_{xi}t$ or $x_f - x_i = v_{xi}t$ displacement = velocity × time	$y_f = y_i + v_{yi}t - \frac{1}{2}gt^2$ Here we have an accelerated motion.

Conclusion: We divide a two-dimensional motion into TWO one-dimensional motion

Some calculations on Projectile motion

Need to calculate:

* Maximum height
* Time of flight
* Range of the projectile



* At the maximum height $v_y = 0$, $y = y_{max}$

$\Rightarrow$

$$0 = v_{yi} - g\,t_{max} \qquad t_{max} = \text{time to reach maximum height}$$

$\Rightarrow$

$$t_{max} = \frac{v_{yi}}{g}$$

Now we use $y_f = y_i + v_{yi}t - \frac{1}{2}gt^2$ to find y_{max}

$$y_{max} = 0 + v_{yi}\,t_{max} - \frac{1}{2}g\,t_{max}^2$$

$$= v_{yi}\frac{v_{yi}}{g} - \frac{1}{2}g\frac{v_{yi}^2}{g^2}$$

$\Rightarrow$

$$\boxed{y_{max} = \frac{1}{2}\frac{v_{yi}^2}{g}}$$

* To find the time of flight, use

$$y_f = y_i + v_{yi}t - \frac{1}{2}gt^2$$

$\Rightarrow$

$$y_f = 0 \;,\; y_i = 0 \qquad t = t_{flight}$$

$$\Rightarrow \quad 0 = 0 + v_{yi}\,t_{flight} - \frac{1}{2}g\,t_{flight}^2$$

$\Rightarrow$

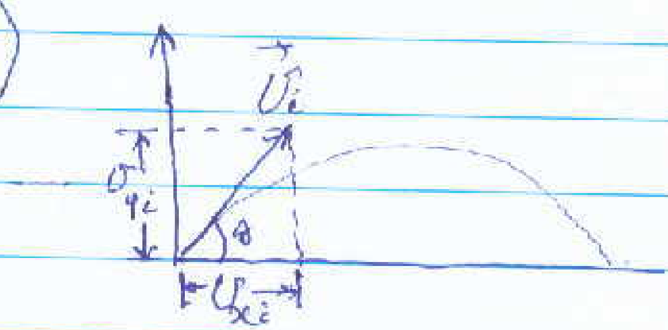
$$0 = \left(v_{yi}\,t_{flight} - \frac{1}{2}g\,t_{flight}^2\right)$$
$$= \left(v_{yi} - \frac{1}{2}g\,t_{flight}\right)t_{flight}$$

$$\Rightarrow \boxed{t_{flight} = \frac{2v_{yi}}{g}}$$

* To find the range x_{max}

$$x_{max} = v_{xi}\, t_{flight}$$

$$\Rightarrow \boxed{x_{max} = \frac{2 v_{xi} v_{yi}}{g}}$$



But

$$v_{xi} = v_i \cos\theta$$

$$v_{yi} = v_i \sin\theta$$

$$\Rightarrow$$

$$x_{max} = \frac{2(v_i \cos\theta)(v_i \sin\theta)}{g}$$

$$= \frac{(v_i^2)(2\sin\theta\cos\theta)}{g}$$

$$\boxed{x_{max} = \frac{v_i^2 \sin(2\theta)}{g}}$$

Also

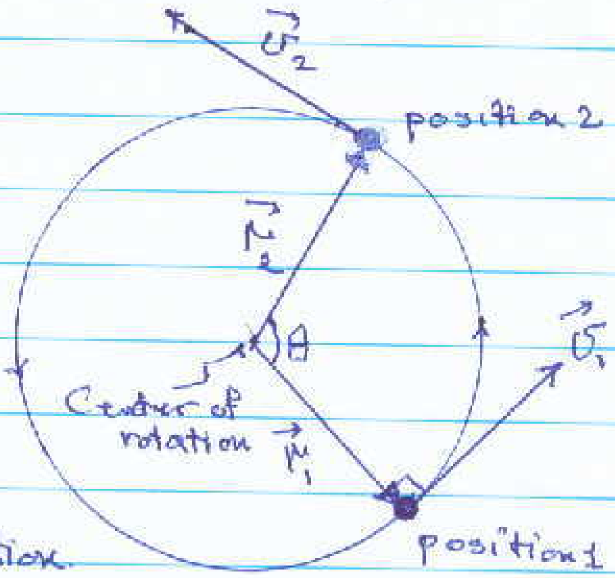
$$\boxed{y_{max} = \frac{v_i^2 \sin^2\theta}{2g}}$$

$$\boxed{t_{flight} = \frac{2 v_i \sin\theta}{g}}$$

Uniform Circular Motion:

Here we study the motion of a body that is going in a circle at constant speed

Acceleration means a change in the rate of $\vec{v}$. i.e. $\vec{a} = \frac{d\vec{v}}{dt}$



In free fall $\Rightarrow \vec{v}$ changes magnitude but not direct $\Rightarrow$ acceleration.

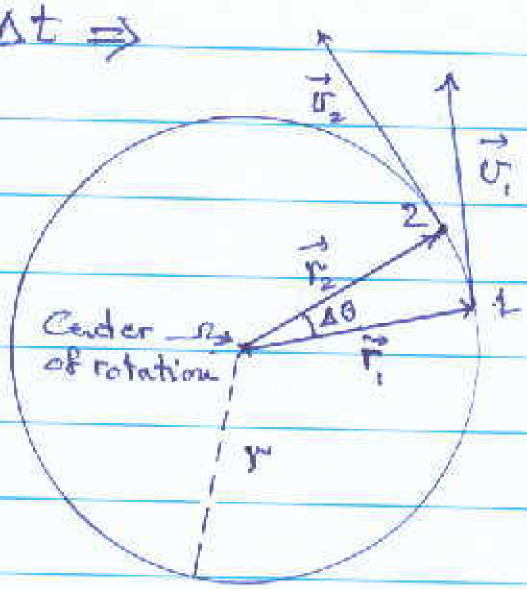
In uniform circular motion $\Rightarrow \vec{v}$ changes direction but not magnitude $\Rightarrow$ acceleration

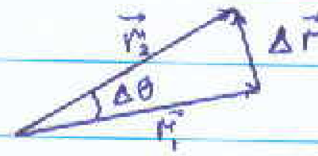
In projectile motion $\Rightarrow \vec{v}$ changes magnitude and direction $\Rightarrow$ acceleration

Therefore in uniform circular motion $\Rightarrow |\vec{r}_1| = |\vec{r}_2|$, $|\vec{v}_1| = |\vec{v}_2|$, but $\vec{v}$ changes direction.

Consider a small change in position $\Rightarrow \Delta\theta$ in a small time interval Δt.

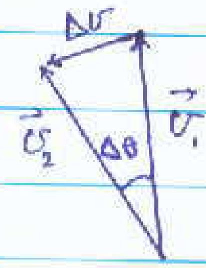
In $\Delta t \Rightarrow$





$\vec{r}_2 = \vec{r}_1 + \Delta\vec{r}$

$|\vec{r}_2| = |\vec{r}_1| = r$



$\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$

$|\vec{v}_2| = |\vec{v}_1| = v$

$\Rightarrow$ We have two similar triangles that have the same vertix angle $\Delta\theta$

$\Rightarrow$

$$\frac{\Delta r}{r} \simeq \Delta\theta \quad , \quad \frac{\Delta v}{v} = \Delta\theta$$

$\Rightarrow$

$$\frac{\Delta r}{r} = \frac{\Delta v}{v}$$

or

$$\Delta v = \frac{v}{r}\Delta r$$

Divide by Δt

$\Rightarrow$

$$\frac{\Delta v}{\Delta t} = \frac{v}{r}\frac{\Delta r}{\Delta t}$$

Take limit when $\Delta t \to 0$

$\Rightarrow$

$$\frac{dv}{dt} = \frac{v}{r}v = \frac{v^2}{r}$$

or

$$\boxed{a = \frac{v^2}{r}}$$

. The direction of $\Delta\vec{v}$ when $\Delta t \to 0$

$\Rightarrow \Delta\theta \to 0 \Rightarrow \Delta\vec{v}$ is $\perp$ to $\vec{v}$.

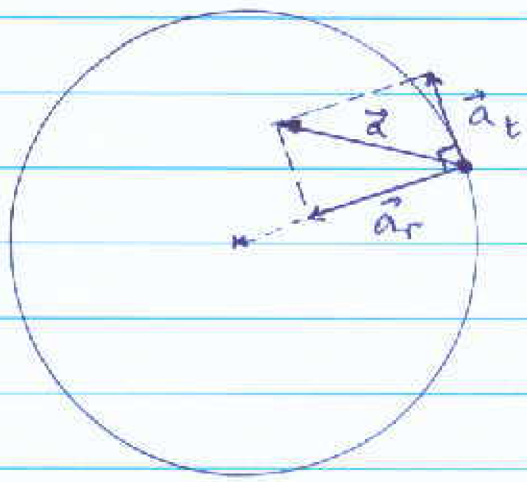
$\Rightarrow$ <u>Direction of $\vec{a}$ is towards the center of rotation.</u>

$\Rightarrow \vec{a} = \vec{a}_r$ (radial acceleration)

$$\boxed{a = \frac{v^2}{r}}$$

Nonuniform Circular Motion:

Here $\vec{v}$ changes magnitude and direction, i.e. the speed of the rotating body changes along the circular path.



The result is a tangential acceleration $\vec{a}_t$ in the direction of the tangent (التعجيل المماسي)

In such a case $\vec{a}$ = total acceleration

$$\boxed{\vec{a} = \vec{a}_t + \vec{a}_r}$$