Best $p$-Simultaneous Approximation in Some Metric Space

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Abstract

Let $X$ be a Banach space, $(I, \mu)$ be a finite measure space, and $\Phi$ be an increasing subadditive continuous function on $[0, +\infty)$ with $\Phi(0) = 0$. In the present paper, we discuss the best $p$-simultaneous approximation of $L^\Phi(I, G)$ in $L^\Phi(I, X)$ where $G$ is a closed subspace of $X$.

Key Words: Simultaneous, Approximation.

1. Introduction

A function $\Phi : [0, +\infty) \to [0, +\infty)$ is called a modulus function if it satisfies the following conditions:

1. $\Phi(x) = 0$ iff $x = 0$.
2. $\Phi(x + y) \leq \Phi(x) + \Phi(y)$.
3. $\Phi$ is a continuous increasing function.

For a modulus function $\Phi$, a finite measure space $(I, \mu)$ and a Banach space $X$,

\[ L^\Phi(I, X) = \{ f : I \to X : \int_I \Phi(\|f(t)\|) \, d\mu < +\infty \}. \]

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For \( f \in L^\Phi(I, X) \), define
\[
\|f\|_\Phi = \int_I \Phi(\|f(t)\|)d\mu.
\]
In fact \((L^\Phi(I, X), \| \cdot \|_\Phi)\) is a complete metric linear space \([4]\). Further, it is known that \(L^1(I, X) \subseteq L^\Phi(I, X)\). For more information about \(L^\Phi(I, X)\), we refer to \([3, 5]\). For \(x_1, x_2\) in \(X\) and \(1 < p < +\infty\), we set
\[
|(x_1, x_2)|_{\Phi, p} = ((\Phi(\|x_1\|))^p + (\Phi(\|x_2\|))^p)^{\frac{1}{p}}.
\]
Note that \((X^2, \cdot |_{\Phi, p})\) is a complete metric space. The diagonal of \(G^2\) is given by \(D = \{(g, g) : g \in G\}\). Throughout this paper, \(X\) is a Banach space, \(G\) is a closed subspace of \(X\) and \(\Phi\) is a modulus function. For \(f_1\) and \(f_2\) in \(L^\Phi(I, X)\), we set
\[
|(f_1, f_2)|_{\Phi, p} = \|f_1\|_\Phi^p + \|f_2\|_\Phi^p,
\]
for all \(1 < p < +\infty\). Then \((L^\Phi(I, X))^2, \cdot |_{\Phi, p})\) is a complete metric space. We consider \(X\) as a metric space with a metric \(d(x, y) = \Phi(\|x - y\|)\).

**Definition 1.1** For \(x_1, x_2 \in G\), define \(\text{dist}_\Phi : X^2 \to \mathbb{R}\) by
\[
\text{dist}_\Phi(x_1, x_2, G) := \inf_{z \in G} \left[ (\Phi(\|x_1 - z\|))^p + (\Phi(\|x_2 - z\|))^p \right]^{\frac{1}{p}}.
\]
Consequently, for \(f_1, f_2 \in L^\Phi(I, X)\), we define
\[
\text{dist}_\Phi(f_1, f_2, L^\Phi(I, X)) := \inf_{g \in L^\Phi(I, G)} \left[ \|f_1 - g\|_\Phi^p + \|f_2 - g\|_\Phi^p \right]^{\frac{1}{p}}.
\]

**Definition 1.2** We say that \(z \in G\) is a best \(p\)-simultaneous approximation from \(G\) of pair elements \(x_1, x_2 \in X\) if
\[
[(\Phi(\|x_1 - z\|))^p + (\Phi(\|x_2 - z\|))^p]^\frac{1}{p} \leq [(\Phi(\|x_1 - y\|))^p + (\Phi(\|x_2 - y\|))^p]^\frac{1}{p}
\]
for every \(y \in G\). We say that \(g \in L^\Phi(I, G)\) is the best \(p\)-simultaneous approximation of a pair of elements \(f_1, f_2\) in \(L^\Phi(I, X)\), if for every \(h \in L^\Phi(I, G)\), we have
\[
\|f_1 - g\|_\Phi^p + \|f_2 - g\|_\Phi^p \leq \|f_1 - h\|_\Phi^p + \|f_2 - h\|_\Phi^p.
\]
Note that for \( g \in G \) is the best \( p \)-simultaneous approximation from \( G \) of \( x_1, x_2 \in X \) iff \((g, g)\) is the best approximation from \( D \) of the pair \((x_1, x_2)\) \( X^2 \) where the metric on \( X^2 \) is \(| \cdot |_{\Phi^p} \). If every pair of elements \( x_1, x_2 \in X \) admits a best \( p \)-simultaneous approximation from \( G \), then \( G \) is said to be \( p \)-simultaneous proximinal in \( X \). The problem of best simultaneous approximation has been studied by many authors in [2, 7, 12, 13]. Most of these works have dealt with the characterization of best simultaneous approximation in space of continuous functions with values in a Banach space \( X \). Results of best simultaneous approximation in general Banach space can be found in [1, 8, 10]. Some results were obtained in the spaces of \( L^p(I, X) \) have been tackled in [6, 11]. In the present paper, we investigate the best \( p \)-simultaneous approximations of \( L^p(I, G) \) in \( L^p(I, X) \) where \( G \) is a closed subspace of \( X \).

2. Main Results

We start with the following technical lemma.

**Lemma 2.1** Suppose \( 1 < p < +\infty \). For \( f_1, f_2 \in L^p(I, X) \) we have

1. \[ \int_I \text{dist}_\Phi(f_1(t), f_2(t), G) d\mu \leq 2 \text{dist}_\Phi(f_1, f_2, L^p(I, G)). \]

2. \[ \text{dist}_\Phi(f_1, f_2, L^p(I, G)) \leq \int_I \text{dist}_\Phi(f_1(t), f_2(t), G) d\mu \]

**Proof.** For any \( g \in L^p(I, G) \) and \( t \in I \), we have

\[ [\text{dist}_\Phi(f_1(t), f_2(t), G)]^p \leq [\Phi||f_1(t) - g(t)||]_p^p + [\Phi||f_2(t) - g(t)||]_p^p \leq [\Phi||f_1(t) - g(t)|| + \Phi||f_2(t) - g(t)||]_p^p. \]

Therefore we have,

\[ \int_I \text{dist}_\Phi(f_1(t), f_2(t), G) d\mu \leq ||f_1 - g||_\Phi + ||f_2 - g||_\Phi \]

\[ \leq 2 ||f_1 - g||_\Phi^p + ||f_2 - g||_\Phi^p. \]

After taking the infimum over all \( g \) in \( L^p(I, G) \), we finish our proof of inequality (1). By the density of simple functions in \( L^p(I, X) \), we have for any \( \varepsilon > 0 \) there are two simple functions \( f_1^* \) and \( f_2^* \) in \( L^p(I, X) \) such that

\[ ||f_1^* - f_1||_\Phi \leq \frac{\varepsilon}{2^p+1}, \]
and
\[ ||f_1^* - f_2||_\Phi \leq \frac{\varepsilon}{2^{p+1}}. \]

We can write \( f_i^* = \sum_{k=1}^{n} \chi_{A_k} x_k^1, \) where \( A_k, k = 1, 2, \ldots n \) are disjoint measurable subsets of \( I \) satisfying \( \bigcup_{k=1}^{n} A_k = I \) and \( \chi_{A_k} \) is the characteristic function of \( A_k, \) and \( x_k^1 \in X. \) We may assume that \( \mu(A_k) > 0, \) for \( k = 1, 2, \ldots n. \) Since
\[ \text{dist}_\Phi(x, y, G) = \inf_{z \in G} [(\Phi(||x - z||))^p + (\Phi(||y - z||))^p]^{\frac{1}{p}}, \]
then for any \( k > 0, \) we can select \( y_k \in G \) such that
\[ [(\Phi(||x_k^1 - y_k||))^p + (\Phi(||x_k^2 - y_k||))^p]^{\frac{1}{p}} < \text{dist}_\Phi(x_k^1, x_k^2, G) + \frac{\varepsilon}{2n\mu(A_k)}. \]

Let \( g = \sum_{k=1}^{n} \chi_{A_k} y_k. \) Clearly \( g \in L^\Phi(I, G). \) Now
\[
\text{dist}_\Phi(f_1, f_2, L^\Phi(I, G)) \leq ||f_1 - g||_\Phi^p + ||f_2 - g||_\Phi^p
\leq [(||f_1 - f_1^*||_\Phi + ||f_1^* - g||_\Phi)^p + (||f_2 - f_2^*||_\Phi + ||f_2^* - g||_\Phi)^p]^{\frac{1}{p}}
\leq ||f_1 - f_1^*||_\Phi^p + ||f_2 - f_2^*||_\Phi^p + ||f_1^* - g||_\Phi^p + ||f_2^* - g||_\Phi^p.
\]

It is easy to show that:
\[
\text{dist}_\Phi(f_1, f_2, L^\Phi(I, G)) < \frac{\varepsilon}{2} + \left[ \left( \int_I \Phi(||f_1^*(t) - g(t)||) \, d\mu \right)^\frac{1}{p} + \left( \int_I \Phi(||f_2^*(t) - g(t)||) \, d\mu \right)^\frac{1}{p} \right]^p
\leq \frac{\varepsilon}{2} + \sum_{k=1}^{n} \mu(A_k) \left[ (\Phi(||x_k^1 - y_k||))^p + (\Phi(||x_k^2 - y_k||))^p \right]^{\frac{1}{p}}
\leq \frac{\varepsilon}{2} + \sum_{k=1}^{n} \mu(A_k) \left[ \text{dist}_\Phi(x_k^1, x_k^2, G) + \frac{\varepsilon}{2n\mu(A_k)} \right].
\]
Thus
\[
\text{dist}_\Phi(f_1, f_2, L^\Phi(I, G)) < \varepsilon + \int_I \text{dist}_\Phi(f_1^*(t), f_2^*(t), G) \, d\mu. \quad (1)
\]
Using the subadditivity of $\Phi$, we have

\[
dist_\Phi(f_1^*(t), f_2^*(t), G) \leq \dist_\Phi(f_1(t), f_2(t), G) + \left[\Phi\left(||f_1^*(t) - f_1(t)||\right)\right]^p + \left[\Phi\left(||f_2^*(t) - f_2(t)||\right)\right]^p,
\]

for all $t$. Therefore

\[
\int_I \dist_\Phi(f_1^*(t), f_2^*(t), G) d\mu \leq \int_I \dist_\Phi(f_1(t), f_2(t), G) d\mu + \left[\Phi\left(||f_1^*(t) - f_1(t)||\right)\right]^p + \left[\Phi\left(||f_2^*(t) - f_2(t)||\right)\right]^p.
\]

Thus

\[
\dist_\Phi(f_1, f_2, L^\Phi(I, G)) \leq \int_I \dist_\Phi(f_1(t), f_2(t), G) d\mu + \frac{\epsilon}{2^p},
\]

and hence

\[
\dist_\Phi(f_1, f_2, L^\Phi(I, G)) \leq \int_I (\dist_\Phi(f_1(t), f_2(t), G)) d\mu.
\]

\[\square\]

**Lemma 2.2** Suppose that $1 < p < +\infty$. If $f_1, f_2$ are simple functions in $L^\Phi(I, X)$ and if $\mu(I) = 1$, then

\[
\dist_\Phi(f_1, f_2, L^\Phi(I, G)) = \left[\int_I \dist_\Phi(f_1(t), f_2(t), G)^p d\mu\right]^{\frac{1}{p}}.
\]

**Proof.** By Inequality (2) of Lemma 2.1, and the assumption $\mu(I) = 1$, we have

\[
\dist_\Phi(f_1, f_2, L^\Phi(I, G)) \leq \int_I \dist_\Phi(f_1(t), f_2(t), G) d\mu \leq \left[\int_I \dist_\Phi(f_1(t), f_2(t), G)^p d\mu\right]^{\frac{1}{p}}.
\]

Given $\epsilon > 0$. Choose $g_0 \in L^\Phi(I, G)$ with the property that

\[
||f_1 - g_0||_\Phi^p + ||f_2 - g||_\Phi^p < (\dist_\Phi(f_1, f_2, L^\Phi(I, G))^p + \epsilon.
\]

(2)

Since $f_1, f_2$ are simple functions in $L^\Phi(I, X)$, we let

\[
f_i(t) = \sum_{k=1}^n x^i_k \chi_{A_k}(t), \quad (i = 1, 2)
\]
where $A_k, k = 1, 2, \ldots, n$ are disjoint measurable sets with $\mu(A_k) > 0$, and $x^i_k \in X$. For a simple function $g$ in $L^p(I, G)$, we can set $g(t) = \sum_{k=1}^n y_k \chi_{A_k}(t)$. Thus
\[
\int_I (\text{dist}_\Phi(f_1(t), f_2(t), G))^p d\mu \leq \int_I (\Phi ||f_1(t) - g(t)||)^p d\mu + \int_I (\Phi ||f_2(t) - g(t)||)^p d\mu
\]
\[
= \sum_{k=1}^n \int_{A_k} (\Phi ||x^1_k - y_k||)^p d\mu + \sum_{k=1}^n \int_{A_k} (\Phi ||x^2_k - g(t)||)^p d\mu(t)
\]
\[
= \sum_{k=1}^n \mu(A_k)(\Phi ||x^1_k - y_k||)^p + \sum_{k=1}^n \mu(A_k)(\Phi ||x^2_k - g(t)||)^p
\]
\[
\leq \left[ \sum_{k=1}^n \mu(A_k) \Phi ||x^1_k - y_k|| \right]^p + \left[ \sum_{k=1}^n \mu(A_k) \Phi ||x^2_k - g(t)|| \right]^p
\]
\[
= \left[ \int_I \Phi ||f_1(t) - g(t)|| d\mu \right]^p + \left[ \int_I \Phi ||f_2(t) - g(t)|| d\mu \right]^p
\]
\[
= ||f_1 - g||^p_\Phi + ||f_2 - g||^p_\Phi.
\]
By (2) and the fact that simple functions are dense in $L^p(I, X)$, we have
\[
\int_I (\text{dist}_\Phi(f_1(t), f_2(t), G))^p d\mu \leq ||f_1 - g_0||^p_\Phi + ||f_2 - g_0||^p_\Phi < (\text{dist}_\Phi(f_1, f_2, L^p(I, G))^p + \epsilon.
\]
Since $\epsilon$ is arbitrary, we have
\[
\left[ \int_I (\text{dist}_\Phi(f_1(t), f_2(t), G))^p d\mu \right]^\frac{1}{p} \leq \text{dist}_\Phi(f_1, f_2, L^p(I, G)).
\]

\[\Box\]

**Theorem 2.1** Suppose $\mu(I) = 1$. If $G$ is $p$-simultaneously proximinal in $X$, then for every pair of simple functions $f_1, f_2$ in $L^p(I, X)$, there exists $g \in L^p(I, X)$ such that $g$ is the best simultaneous approximation of the pair of elements $f_1$ and $f_2$.

**Proof.** We can write $f_i = \sum_{k=1}^n \chi_{A_k} x^i_k$, $i = 1, 2$, where $A_k, k = 1, 2, \ldots, n$ are disjoint measurable sets such that $\bigcup_{k=1}^n A_k = I$ with $\mu(A_k) > 0$ for $k = 1, 2, \ldots, n$. Pick
y_k \in G$ such that $y_k$ is the best approximation of the pair of elements $x^1_k, x^2_k \in X$. Let $g = \sum_{k=1}^n \chi_{A_k} y_k$. Then

$$
||f_1 - g||_\Phi^p + ||f_2 - g||_\Phi^p \overset{p}{=} \left[ \int_I \Phi(||f_1(t) - g(t)||)d\mu \right]^p + \left[ \int_I \Phi(||f_2(t) - g(t)||)d\mu \right]^p
$$

$$
= \left[ \sum_{k=1}^n \mu(A_k) \Phi(||x^1_k - y_k||) \right]^p + \left[ \sum_{k=1}^n \mu(A_k) \Phi(||x^2_k - y_k||) \right]^p
$$

$$
\leq \sum_{k=1}^n \mu(A_k) \left[ \Phi(||x^1_k - y_k||)^p + \Phi(||x^2_k - y_k||)^p \right]^\frac{1}{p}
$$

$$
= \sum_{k=1}^n \mu(A_k) \text{dist}_\Phi(x^1_k, x^2_k, G)
$$

$$
= \int_I \text{dist}_\Phi(f_1(t), f_2(t), G)d\mu
$$

$$
\leq \left[ \int_I \text{dist}_\Phi(f_1(t), f_2(t), G)^p d\mu \right]^{\frac{1}{p}}.
$$

Using Lemma 2.2, we have

$$
(||f_1 - g||_\Phi^p + ||f_2 - g||_\Phi^p)^\frac{1}{p} = \text{dist}_\Phi(f_1, f_2, L^\Phi(I, G)).
$$

\[ \square \]

**Theorem 2.2** Let $g \in L^\Phi(I, G)$ be a best $p$-simultaneous approximation of a pair $f_1, f_2 \in L^\Phi(I, X)$. Then for every measurable subset $A$ of $I$ and every $h \in L^\Phi(I, G)$, we have

$$
\int_A \Phi(||f_i(t) - g(t)||)d\mu \leq \int_A \Phi(||f_i(t) - h(t)||)d\mu,
$$

for some $i \in \{1, 2\}$.

**Proof.** Assume that $\mu(A) > 0$ for some $A \subseteq I$. If there is $h_0 \in L^\Phi(I, G)$ doesn’t satisfy (3) for $i = 1, 2$, then we define $g_0 \in L^\Phi(I, G)$ by $g_0(t) = g(t)$ if $t \in I - A$ and $h_0(t)$
if $t \in A$. Thus for $i = 1, 2$ we have
\begin{align*}
    \int_I \Phi(||f_i(t) - g_0(t)||)d\mu &= \int_A \Phi(||f_i(t) - h_0(t)||)d\mu + \int_{I-A} \Phi(||f_i(t) - g(t)||)d\mu \\
    &< \int_I \Phi(||f_i(t) - g(t)||)d\mu.
\end{align*}
Hence we have
\begin{align*}
    ||f_i - g_0||_\Phi^p < ||f_i - g||_\Phi^p, \quad i = 1, 2.
\end{align*}
This contradicts the fact that $g$ is best $p$-simultaneous approximation from $L^p(I, G)$ of a pair of the elements $f_1, f_2$.

**Corollary 2.1** If $g$ is a best $p$-simultaneous approximation from $L^p(I, G)$ of a pair of elements $f_1, f_2 \in L^p(I, G)$, then for every measurable subset $A$ of $I$,
\begin{align*}
    \int_A \Phi(||g(t)||)dp \leq 2\max\left\{\int_A \Phi(||f_1(t)||)dp, \int_A \Phi(||f_2(t)||)dp\right\}.
\end{align*}

**Proof.** It follows from Theorem 2.2 by taking $h = 0$. □

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**References**


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