FULLY $\lambda(P)$-BASE IN A COMPLETE NUCLEAR SPACE

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Abstract

We define a fully $\lambda(P)$-base of type $\lambda(P) - G_\infty$ and of type $\lambda(P) - G_1$. Also, we prove that if $X$ is a complete nuclear space with fully $\lambda(P)$-base which is of type $\lambda(P) - G_\infty$ or of type $\lambda(P) - G_1$, then all fully $\lambda(P)$-bases are quasisimilar.

1. Basic Concepts

In the sequel $X$ will denote an infinite dimensional complete locally convex space. By a seminorm on $X$, continuous seminorm will be meant.

Let $(x_n : f_n)$ be a Schauder base in $X$. A seminorm $p$ on $X$ is called $(x_n)$-normal, if $p(f_n(x)x_n) \leq p(x)$. A sequence $(x_n)$ of vectors in $X$ is called a basic sequence, if there is a closed subspace $Y$ of $X$ such that $(x_n)$ is a base of $Y$, and $(x_n)$ is a complemented basic sequence, if the subspace $Y$ is complemented in $X$. Basic sequences $(x_n)$ and $(y_n)$ are similar, if the convergence of the series $\sum_n t_nx_n$ implies that of $\sum_n t_ny_n$ and vice versa. Basic sequences $(x_n)$ and $(y_n)$ are semisimilar, if there exist

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scalars \( a_n \) making \((a_n x_n)\) and \((a_n y_n)\) similar. \((x_n)\) and \((y_n)\) are quasisimilar, if there are permutations \((r_n)\) and \((s_n)\) of positive integers such that \((x_{r_n})\) and \((y_{s_n})\) are similar.

A base \((x_n : f_n)\) in \(X\) is called equicontinuous, if for each seminorm \(p\) on \(X\) there is a seminorm \(q\) on \(X\) such that

\[
\sup_n |f_n(x)| p(x_n) \leq q(x)
\]

for all \(x \in X\).

A base \((x_n : f_n)\) in \(X\) is called an absolutely equicontinuous basis, if for each seminorm \(p\) on \(X\) there is a seminorm \(q\) on \(X\) with

\[
\sum_n |f_n(x)| p(x_n) \leq q(x),
\]

for all \(x \in X\).

A Köthe set \(P\) will be called a power set of infinite type [5] if it satisfies the following conditions:

1. For each \(a = (a_n) \in P\), \(0 < a_n \leq a_{n+1}\) for all \(n\).
2. For each \(a = (a_n) \in P\), there exists \(b = (b_n) \in P\) such that \(a_n^2 = O(b_n)\).

A Köthe set \(Q\) will be called a power set of finite type [5] if it satisfies the following conditions:

1. Each \(q = (q_n) \in Q\) is a positive non-increasing sequence.
2. For each \(q = (q_n) \in Q\) there exists \(p = (p_n) \in Q\) with \(\sqrt{q_n} = O(p_n)\).

The following results are well known.

**Theorem 1.1** [4]. Every equicontinuous basis of a nuclear space \(X\) is absolutely equicontinuous.

**Proposition 1.1** [1]. If \((y_n)\) is an equicontinuous basic sequence in a nuclear space \(X\), then for every semi-norm \(p\) on \(X\), there is a semi-norm \(q\) on \(X\) with \(p(x) \leq q(x)\) and \(\sum_{n=1}^{\infty} \frac{p(y_n)}{q(y_n)} < +\infty\).
**Proposition 1.2** [1]. If $X$ is a complete nuclear space with an equicontinuous basis $(x_n)$ with coefficient functionals $f_n$, then $X$ is isomorphic to a Köthe space $\lambda(P)$, where

$$P = \{(p_i(x_n)) : i \in I\}$$

and $\{p_i : i \in I\}$ is a complete system of seminorms on $X$.

**Definition 1.1** [1]. A basic sequence $(x_n)$ in a complete nuclear locally convex space $X$ is called of type $G_1$ (resp. of type $G_\infty$), if there is a complete system $\{p_i : i \in I\}$ of $(x_n)$-normal seminorms on the subspace $Y = \text{span}\{x_n\}$ such that the Köthe set

$$P = \{(p_i(x_n)) : i \in I\}$$

is a power set of finite type (resp. of infinite type).

**Definition 1.2** [3]. A Schauder base $(x_n : f_n)$ for a locally convex space $X$ is called

1. a semi-$\lambda(P)$-base, if for every seminorm $p$ on $X$, the map $\psi_p : X \rightarrow \lambda(P)$ is well defined, where $\psi_p(x) = (p(x_n)f_n(x))$, $x \in X$.

2. a fully $\lambda(P)$-base, if it is a semi-$\lambda(P)$-base and for each a seminorm $p$ on $X$, $\psi_p : X \rightarrow \lambda(P)$ is continuous.

The following results are well known.

**Proposition 1.3** [3]. Let $X$ be a complete locally convex space having a fully $\lambda(P)$-base $(x_n : f_n)$. Then $X$ can be identified with a Köthe space $\lambda(P_0)$, where the Köthe set $P_0$ is given by

$$P_0 = \{(p(x_n)b_n) : p \text{ is a seminorm on } X, b \in P\}.$$ 

2. **Main Results**

Now, we introduce the following definitions in order to facilitate our subsequent arguments.
**Definition 2.1.** A sequence \((x_n)\) of vectors in a locally convex space \(X\) is called a **fully \(\lambda(P)\)-basic sequence**, if there is a closed subspace \(Y\) of \(X\) such that \((x_n : f_n)\) is a fully \(\lambda(P)\)-base of \(Y\), and \((x_n)\) is a **complemented \(\lambda(P)\)-basic sequence**, if the subspace \(Y\) is complemented in \(X\).

**Definition 2.2.** A fully \(\lambda(P)\)-base \((x_n : f_n)\) in a complete locally convex space \(X\) is called of **type \(\lambda(P)\) \(G_1\)** (resp. of **type \(\lambda(P)\) \(G_\infty\)**), if there is a complete system \(\{p_i : i \in I\}\) of \((x_n)\)-normal seminorms on the subspace \(Y = \overline{\text{span}\{x_n\}}\) such that the Köthe set
\[
P_0 = \{(p_i(x_n)b_n) : i \in I, b \in P\}
\]
is a power set of finite type (resp. of infinite type). Such a system of seminorms will be called **admissible**, where the set \(I\) is a partially ordered set which is directed upwards.

The following result is crucial in our work.

**Proposition 2.1** [5]. Let \(\lambda(P)\) and \(\lambda(Q)\) be nuclear smooth sequence spaces of infinite and finite type, respectively. Then

1. \(\Delta(\lambda(P)) = \lambda'(P)\).
2. \(\Delta(\lambda(Q)) = \lambda(Q)\).

Let \(P'\) be any Köthe set with the following additional condition: For each \(a \in P'\), we have \(\inf\{a_n : n \in \mathbb{N}\} > 0\).

**Proposition 2.2.** Every fully \(\lambda(P')\)-base \((x_n : f_n)\) in a complete locally convex space \(X\) is a fully \(\ell_1\)-base.

**Proof.** Note that \(\ell_1\) is a Köthe space generated by the Köthe set \(P = \{(1, 1, \ldots), (2, 2, \ldots), \ldots\}\). Let \(p\) be any seminorm on \(X\). Then for \(m \in \mathbb{N}\) and \((t_n) \in P'\), we have
\[
\sum_n |f_n(x)|p(x_n)m \leq \frac{m}{\alpha} \sum_n |f_n(x)|p(x_n)t_n = \frac{m}{\alpha} p_\lambda(\psi_\lambda(x)),
\]
where \(\alpha = \inf\{t_n : n \in \mathbb{N}\}\). Since \((x_n : f_n)\) is a fully \(\lambda(P')\)-base, there is
a seminorm $q$ on $X$ such that $p_i(\psi_p(x)) \leq q(x)$. Therefore

$$\sum_n |f_n(x)|p(x_n)m \leq \frac{m}{a} q(x).$$

So $(x_n : f_n)$ is a fully $\ell_1$-base.

**Proposition 2.3.** A Schauder base $(x_n : f_n)$ for a complete locally convex space $X$ is an absolutely equicontinuous base if and only if its a fully $\ell_1$-base.

**Proof.** ($\Leftarrow$) Clear.

($\Rightarrow$) Assume that $(x_n : f_n)$ be an absolutely equicontinuous base for $X$. Then for each seminorm $p$ on $X$, there is a seminorm $q$ on $X$ such that

$$\sum_n |f_n(x)|p(x_n) \leq q(x).$$

So for each $m \in N$ and for each seminorm $p$ on $X$, there is a seminorm $q$ on $X$ such that

$$\sum_n |f_n(x)|p(x_n)m \leq mq(x).$$

Since $\ell_1$ is a Köthe space generated by the Köthe set

$$P = \{(1,1,\ldots), (2,2,\ldots), (3,3,\ldots), \ldots\},$$

$\psi_p : X \to \ell_1$ is continuous. Therefore $(x_n : f_n)$ is a fully $\ell_1$-base.

**Proposition 2.4.** If $(x_n : f_n)$ and $(y_n : g_n)$ are fully $\lambda(P')$-bases of a complete nuclear space $X$, both of type $\lambda(P') - G_1$ or of type $\lambda(P') - G_\infty$, then they are similar.

**Proof.** There are admissible systems of seminorms $\{p_i : i \in I\}$ and $\{q_i : i \in J\}$ on $X$ such that

$$P_0 = \{(p_i(x_n)b_n) : i \in I, b \in P'\} \text{ and } Q_0 = \{(q_i(y_n)b_n) : i \in J, b \in P'\}$$

are power sets both of finite type or of infinite type. Then $\lambda(P_0) \cong X \cong \lambda(Q_0)$. Since $X$ is complete it sufficient to show that $\lambda(P_0) = \lambda(Q_0)$. If $(x_n : f_n)$ and $(y_n : g_n)$ are of type $\lambda(P') - G_1$, then
\[ \lambda(P_0) = \Delta(\lambda(P_0)) = \Delta(X) = \Delta(\lambda(Q_0)) = \lambda(Q_0). \]

If \((x_n : f_n)\) and \((y_n : g_n)\) are of type \(\lambda(P') - G_{\infty}\), then by Proposition 2.1,

\[ \lambda'(P_0) = \Delta(\lambda'(P_0)) = \Delta(X) = \Delta(\lambda'(Q_0)) = \lambda'(Q_0). \]

So \(\lambda(P_0) = \lambda(Q_0)\). Therefore \((x_n : f_n)\) and \((y_n : g_n)\) are similar.

**Proposition 2.5.** If \((y_n : h_n)\) is a fully \(\lambda(P')\)-complemented basic sequence in a complete locally convex space \(X\), then there is a system \((g_n)\) of continuous linear functionals on \(X\), \((g_n)\) biorthogonal to \((y_n)\) such that for any seminorm \(p\) on \(X\) and any \(b = (b_n) \in P'\)

\[ q_b(x) = \sup_{n} |g_n(x)| p(y_n) b_n, \]

is a seminorm on \(X\).

**Proof.** Let \(Y = \text{span}\{y_n\}\) be the closed subspace of \(X\) spanned by \((y_n)\), and let \(J : X \to Y\) be a continuous linear projection onto \(Y\). Then \(g_n = h_n J\) is in \(X'\), \(g_n(y_m) = \delta_{nm}\), where \(\delta_{nm} = 1\) if \(n = m\) and \(\delta_{nm} = 0\) if \(n \neq m\). For a seminorm \(p\) on \(X\) and \(b \in P'\), there is \(p'\) on \(Y\) such that

\[ \sup_{n} |h_n(y)| p(y_n) b_n \leq p'(y) \text{ for all } y \in Y. \]

Let \(x \in X\) be given. Then

\[ q(x) = \sup_{n} |g_n(x)| p(y_n) b_n \leq (p' J)x. \]

Since \(p' J\) is a seminorm on \(X\) and \(q(x) \leq (p' J)x\) for all \(x \in X\), \(q\) is also a seminorm on \(X\).

**Proposition 2.6.** If \((y_n : f_n)\) is a fully \(\lambda(P')\)-basic sequence in a complete nuclear space \(X\), then for every seminorm \(p\) on \(X\), there is a seminorm \(q\) on \(X\) with \(p(x) \leq q(x)\) and

\[ \sum_{n} \frac{p(y_n)}{q(y_n)c_n} < +\infty \quad \forall c = (c_n) \in P'. \]

**Proof.** Follows from Propositions 2.2, 2.3 and 1.2.
The proof of the following result follows from the Grothendieck-Pietsch criterion for nuclearity.

**Lemma 2.1.** Suppose that $X$ is a complete nuclear space with a fully $\lambda(P')$-base $(x_n : f_n)$ which is of type $\lambda(P') - G_1$ or of type $\lambda(P') - G_\infty$, then for every seminorm $p$ on $X$ and every $b = (b_n) \in P'$, there are a seminorm $q$ on $X$, $c = (c_n) \in P'$ and a positive constant $\rho > 0$ such that

$$n^2 p(x_n) b_n \leq \rho q(x_n) c_n \quad \text{for all } n \in N.$$

**Lemma 2.2.** Suppose that $X$ is a complete nuclear space with a fully $\lambda(P')$-basis $(x_n : f_n)$ which is of type $\lambda(P') - G_1$ or of type $\lambda(P') - G_\infty$, and $(y_n : g_n)$ is a fully $\lambda(P')$ complemented basic sequence in $X$. Then there are positive integers $k_n$ with $\lim_n k_n = +\infty$ such that $f_{k_n}(y_n) \neq 0$ for all $n \in N$ and such that for any $(x_n)$-normal seminorm $p$ on $X$ and for any $b = (b_n) \in P'$, there are a seminorm $q$ on $X$ and $c = (c_n) \in P'$ such that

$$a_n p(x_{k_n}) \leq p(y_n) \quad \text{and} \quad p(y_n) b_n \leq a_n q(x_{k_n}) c_{k_n},$$

where $a_n = |f_{k_n}(y_n)|$, $n = 1, 2, \ldots$.

**Proof.** Let $P$ be any $(x_n)$-normal seminorm on $X$ and let $b = (b_n)$ be any sequence in $P'$ be given. Then there is a sequence $(g_n)$ in $X'$ with $g_n(y_n) = \delta_{nm}$ such that

$$p(x_n) = \sup_x |g_n(x)| p(y_n) b_n$$

is a seminorm on $X$. Since $g_n$ is linear, we have

$$1 = g_n(y_n) = \sum_{k=1}^{\infty} g_n(x_k) f_k(y_n).$$

Therefore

$$\sum_{k=1}^{\infty} |g_n(x_k)| |f_k(y_n)| \geq 1.$$

Let $\sigma = \sum k^{-2}$. Since $1 = \sigma^{-1} \sum k^{-2}$, we conclude that for each $n$
there is a positive integer \( k_n \) with

\[
\left| g_n(x_{k_n}) \right| \geq \sigma^{-1}k_n^{-2}.
\]

Let \( a_n = |f_{k_n}(y_n)| \). Then

\[
\left| g_n(x_{k_n}) \right| \geq \sigma^{-1}k_n^{-2}a_n^{-1} \quad \text{for all } n \in N. \tag{1}
\]

Let \( t_b = \inf\{b_n : n \in \mathbb{N}\} \). Since \( p' \) is a seminorm on \( X \) and \( (b_n) \in P' \), by Lemma 2.1 and Proposition 2.6, there are a seminorm \( q \) on \( X \) and \( c = (c_n) \in P' \) such that \( p'(x_n) \leq t_b^{-1}p'(x_n)b_n \leq \sigma^{-1}n^{-2}q(x_n)c_n \) and

\[
\sum_n \frac{p(y_n)}{q(y_n)c_n} < +\infty.
\]

Therefore

\[
\left| g_n(x_{k_n}) \right| p(y_n)b_n \leq \sup_j \left| g_j(x_{k_n}) \right| p(y_j)b_j
\]

\[
= p'(x_{k_n}) \leq \sigma^{-1}k_n^{-2}q(x_{k_n})c_{k_n}. \tag{2}
\]

From equations (1) and (2) we have

\[
p(y_n)b_n \leq a_nq(x_{k_n})c_{k_n}. \tag{2.1}
\]

Since \( p \) is \((x_n)\) normal, we have \( a_n p(x_{k_n}) \leq p(y_n) \). Since \( q \) is a seminorm on \( X \) and \( c = (c_n) \in P' \), by inequality 2.1, there are a seminorm \( q' \) on \( X \) and \( c' \in P' \) such that

\[
q(y_n)c_n \leq a_nq'(x_{k_n})c'_{k_n},
\]

also since \( p(y_n) \geq a_n p(x_{k_n}) \), we have

\[
\frac{p(y_n)}{q(y_n)c_n} \geq \frac{p(x_{k_n})}{q'(x_{k_n})c'_{k_n}}.
\]

Hence

\[
\sum_n \frac{p(x_{k_n})}{q'(x_{k_n})c'_{k_n}} < +\infty.
\]
Hence, for any \( j \) for which \( p(x_j) \neq 0 \), the set \( \{ n : k_n = j \} \) must be finite. Since for each \( j \) we can find an \( (x_n) \)-normal seminorm \( p \) on \( X \) for which \( p(x_j) \neq 0 \), all the sets \( \{ n : k_n = j \} \) are finite, that is, \( \lim_n k_n = +\infty \).

**Theorem 2.1.** Suppose that \( X \) is a complete nuclear space with a fully \( \lambda(P') \)-base \( (x_n : f_n) \) of type \( \lambda(P') - G_1 \) or of type \( \lambda(P') - G_\infty \) and \( (y_n : g_n) \) is a fully \( \lambda(P') \)-complemented basic sequence in \( X \). Then there are positive integers \( k_n \) with \( \lim_n k_n = \infty \), positive real numbers \( a_n \) and a complete system \( \{ q_i : i \in I \} \) of \( (y_n) \)-normal seminorms on \( \overline{\text{span}\{y_n : n \in \mathbb{N}\}} \) such that \( q_i(a_n^{-1}y_n) = p_i(x_{k_n}) \), where \( \{ p_i : i \in I \} \) is an admissible system of seminorms on \( X \) for \( (x_n) \).

**Proof.** Let \( (k_n) \), \( (a_n) \) be as in Lemma 2.2. We may assume that \( e = (1, 1, ...) \in P' \). So by Propositions 2.5 and 2.6, \( p_i(y) = \sup_n |g_n(y)|p_i(y_n) \), \( i \in I \) form a complete system for \( Y = \overline{\text{span}\{y_n : n \in \mathbb{N}\}} \). Let \( q_i(y) = \sup_n a_n |g_n(y)|q_i(x_{k_n}) \), \( i \in I \). Then by first inequality of Lemma 2.2, \( \{ q_i : i \in I \} \) is a complete system for \( Y \). Also \( q_i \) is \( (y_n) \)-normal seminorm and \( q_i(a_n^{-1}y_n) = p_i(x_{k_n}) \).

In the rest of the present paper, we assume that \( P' \) satisfies the following additional conditions:

- \((r_1)\) If \( b \in P' \) and \( \pi \) is any permutation on \( \mathbb{N} \), then there are \( \rho > 0 \) and \( c \in P' \) such that \( b_n \leq \rho c_{\pi(n)} \) for all \( n \in \mathbb{N} \).
- \((r_2)\) If \( b \in P' \), and \( (k_n) \) is a nondecreasing sequence, then \( (b_{k_n}) \in P' \).
- \((r_3)\) If \( b \in P' \) and \( (k_n) \) is subsequence of \( (k) \), then there exists \( c = (c_n) \in P' \) such that \( b_n = O(c_{k_n}) \) for all \( k \in \mathbb{N} \).

Now, we have the following:

**Corollary 2.1.** Suppose that \( X \) is a complete nuclear space with a fully \( \lambda(P') \)-basis \( (x_n : f_n) \) which is of type \( \lambda(P') - G_1 \) or of type \( \lambda(P') - G_\infty \). Then every fully \( \lambda(P') \)-complemented basic sequence
(\(y_n : g_n\)) in \(X\) is quasisimilar to a fully \(\lambda(P')\)-complemented basic sequence of the same type of \((x_n : f_n)\).

**Proof.** Let \(\{p_i : i \in I\}\) be an admissible system for \((x_n : f_n)\). Then by Theorem 2.1, there are positive integers \(k_n\) with \(\lim_n k_n = +\infty\), positive real numbers \((a_n)\) and a complete system of \((y_n)\)-normal seminorms \(\{q_i : i \in I\}\) on \(Y = \text{span}\{y_n\}\) such that \(q_i(a_n^{-1}y_n) = p_i(x_{k_n})\).

Since \(\lim_n k_n = +\infty\), there is a permutation \(\pi\) on \(N\) such that \(k_{\pi(n)}\) is non-decreasing. Let \(z_n = a_{\pi(n)}^{-1}y_{\pi(n)}\). Then \(q_i(z_n) = p_i(x_{k_{\pi(n)}})\). Since \((y_n)\) is a fully \(\lambda(P')\)-complemented basic sequence in \(X\) and \(P'\) satisfies condition \((r_1)\), \((z_n)\) is a fully \(\lambda(P')\)-complemented basic sequence in \(X\). Also, since \(\pi\) is a permutation on \(N\) such that \(k_{\pi(n)}\) is non-decreasing and \(P'\) satisfies conditions \((r_2)\) and \((r_3)\), the type of \((z_n)\) is of the same as the type of \((x_n)\). Also \(\sum_n t_n y_{\pi(n)}\) converges if and only if \(\sum_n a_{\pi(n)} f_n z_n\) converges, that is, \((y_n)\) and \((z_n)\) are quasisimilar.

**Theorem 2.2.** If \(X\) is a complete nuclear space with a fully \(\lambda(P')\)-base \((x_n : f_n)\) which is of type \(\lambda(P') - G_1\) or of type \(\lambda(P') - G_\omega\), then all fully \(\lambda(P')\)-bases in \(X\) are quasisimilar.

**Proof.** If \((w_n)\) is an arbitrary fully \(\lambda(P')\) basis in \(X\), then by Corollary 2.1, \((w_n)\) is quasisimilar to a fully \(\lambda(P')\)-base \((y_n)\) of \(X\) and the type of \((y_n)\) is the same of the type of \((x_n)\). So by Proposition 2.4, \((x_n)\) and \((y_n)\) are similar. Therefore \((x_n)\) and \((y_n)\) are quasisimilar.

The following result is consequence of our results.

**Corollary 2.2** [1]. If \(X\) is a complete nuclear locally convex space with an equicontinuous basis \((x_n)\) which is of type \(G_1\) or of type \(G_\omega\), then all equicontinuous bases in \(X\) are quasisimilar.

**Proof.** If \((x_n)\) is an equicontinuous base of \(X\), then by Theorem 1.1, \((x_n)\) is an absolutely equicontinuous base, and hence its a fully \(\ell_1\) -base.
Therefore by Theorem 2.2, all fully $\ell_1$-bases in $X$ are quasisimilar. Therefore, by Proposition 2.3, all equicontinuous bases of $X$ are quasisimilar.

Now, in order to show that our work is meaningful we give an example of a Köthe set $P$ with $\lambda(P) \neq \ell_1$ and $P$ satisfies conditions $(r_1)$, $(r_2)$ and $(r_3)$.

**Example 2.1.** Let $P = \{(a_n) : (a_n) \text{ is positive sequence with } \inf\{a_n : n \in \mathbb{N}\} > 0\}$. Then

1. $P$ satisfies conditions $(r_1)$, $(r_2)$ and $(r_3)$.

2. $\lambda(P) = \varnothing \neq \ell_1$, where $\varnothing = \{(x_n) : x_n = 0 \text{ for all but finitely many}\}$.

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