ME 535 Fall 2012 – Problem Set 1 (Review of ODE Solution Methods)

Due in class Thursday, September 20

1. Use both the separation of variables method and the integrating factor method to solve the ODE,
$$y' + 4y = 8$$

2. Determine the general solution of the following ODEs:
$$\frac{dy}{dx} = \frac{x-y}{x+y}, \qquad \frac{dy}{dx} = \frac{2xy - e^y}{x(e^y - x)}$$

3. Use any available method to solve the following nonhomogeneous ODE problem subject to the initial condition $y(1) = 2$.
$$xy' + y = 6x^2$$

4. Solve the following linear ODE initial value problems:
   (a) $y'' - 4y' - 5y = 0, \quad y(1) = 1, \quad y'(1) = 0$
   (b) $y'' + 6y' + 9y = 0, \quad y(1) = e, \quad y'(1) = -2$

5. Obtain the general solution of the ODE: $y^{iv} - 2y'' - 3y = 0$

6. Obtain the general solution using the method of *Undetermined Coefficients.*
   (a) $y' + 2y = 3e^{2x} + 4\sin x$
   (b) $y'' + y' = 4xe^x + 3\sin x$

7. Obtain the general solution using the method of *Variation of Parameters*,
$$y'' - 2y' + y = 6x^2$$

8. The equation of motion of a body of mass m falling under the action of gravity and subject to an upward aerodynamic drag force $f(v)$ is
$$mv' = mg - f(v)$$
where $v(t)$ is the velocity of the mass as a function of time. First solve the equation for the case where $f(v) = cv$ (c is a constant) with the initial condition $v(0) = 0$. What is the terminal velocity of the mass in terms of the model parameters?

   Next solve the problem with the same initial condition for the case where $f(v) = cv^2$. Use the approach that introduces a change in dependent variable described below. Start by showing how the equation for u is obtained from the original nonlinear equation by the change of variable.

The Riccati equation is an important first-order, nonlinear ODE,

$$y' = p(x)y^2 + q(x)y + r(x).$$

By changing the dependent variable from y to u using

$$y = Y(x) + \frac{1}{u},$$

the following linear ODE is obtained for u:

$$u' + [2p(x)Y(x) + q(x)]u = -p(x).$$

The function $Y(x)$ is a particular solution of the Riccati equation that must be found by trial and error or inspection (e.g., if $r(x)$ is a constant, assume $Y(x)$ is a constant).

Problem Set 2 Engineering Analysis

1. For the Cauchy-Euler equation shown below, use the change of independent variable, $z = \ln x$ to transform the equation into one with constant coefficients. Start by defining $w(z) = y(x(z))$ and then employ the chain rule to obtain the constant coefficient ODE in terms of w.

$$x^2y'' + \alpha xy' + \beta y = 0$$

2. For the ODE below, what are $p(x)$ and $q(x)$? Is the point $x_0 = -5$ a singular point? Determine the recursion formula for a power series solution about that point and calculate the first four nonvanishing terms in the series for one solution, $y_1(x)$.

$$xy'' + y' + y = 0$$

3. For the equations below, identify all singular points (if any) and classify each as regular or irregular. Do not compute the series.

   (a) $(x+1)^2y'' - 4y' + (x+1)y = 0$

   (b) $xy'' + (\sin x)y' - (\cos x)y = 0$

   (c) $(xy')' - 5y = 0$

4. Solve the variable coefficient, linear ordinary differential equations given below using an appropriate analytical technique,

$$x^2y'' + xy' - y = 0$$

$$y'' - 2x^2y' + 8y = 0, \qquad y(0) = 0, \qquad y'(0) = 1$$

5. Obtain the recursive relation for Legendre's equation for $x_0 = 0$.

6. For the differential equation given below, verify that $x = 0$ is a regular singular point and use the method of Frobenius to obtain the general solution $y(x) = Ay_1(x) + By_2(x)$.

$$2xy'' + y' + x^3y = 0$$

7. Use the procedure for reducing ODEs to Bessel's equation presented in class to obtain the solution to the following:

   (a) $xy'' - 2y' - x^2y = 0$

   (b) $xy'' - 2y' + xy = 0$

8. The equation below will be used to study numerical solution methods for ODEs.

$$y' - 2xe^{-y} = 0, \qquad y(1) = -1$$

   (a) Calculate the exact solution to the ODE.

Engineering Analysis - Problem Set 3

1. Solve the ODE boundary values problems (BVPs) below. Here, the subscript x refers to the derivative with respect to x, so u_{xx} is the second derivative. $u_{xx} = \frac{d^2u}{dx^2}$

$$u_{xx} - Hu = 0, \qquad u_x(0) = Q_1, \qquad u(L) = u_2$$

$$u_{xx} - Vu_x = 0, \qquad u(0) = u_1, \qquad u(L) = u_2$$

where all of the parameters in the equations and boundary conditions are constants.

2. For each of the Sturm-Liouville problem given below, write the solution in terms of the eigenfunctions and determine the expression for the eigenvalues, λ.

$$y'' + \lambda y = 0, \qquad y'(-1) = 0, \quad y'(1) = 0, \qquad -1 < x < 1$$

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(L) = 0, \; ; \qquad 0 < x < L$$

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(\pi) + y'(\pi) = 0, \qquad 0 < x < \pi$$

3. For the second Sturm-Liouville problem above, calculate the eigenfunction expansion for $f(x) = x$.

4. Find the first three nonvanishing coefficients in the Legendre polynomial expansion for the following function

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$$

5. Show that the boundary value problem given below is a Sturm-Liouville problem (i.e., what are $p(x)$, $q(x)$ and $w(x)$). Write the solution in terms of the eigenfunctions and obtain an expression for the eigenvalues, λ. By using the change of independent variable $x = \alpha z$, the equation can be put into the exact form of Bessel's equation. The reason for the change of variable is to remove λ by specifying $\alpha^2\lambda = 1$.

$$x^2y'' + xy' + (\lambda x^2 - 9)y = 0, \qquad 0 < x < 1$$

6. Solve the boundary value problem ~~below numerically using ode45. Because the MATLAB ODE solver is designed to solve initial value problems, the shooting method must be employed.~~

$$y'' - 2x^2y' + 8y = 0, \qquad 0 \le x \le 1$$

$$y(x = 0) = 1, \qquad y(x = 1) = -1$$

6 The equation below is derived from Newton's second law of motion and governs the transverse displacement, $y(x)$, of a flexible string hanging vertically from a fixed point located at $x = 0$. If the end of the string is displaced to one side and released it will oscillate from side to side with a displacement pattern that is composed of different modes each having a specific shape $y(x)$ and a temporal frequency denoted by ω.

$$[\rho g(l - x)y']' + \rho\omega^2 y = 0 \qquad (0 \le x \le l)$$

In the equation, the length of the string is l, ρ is the mass per unit length and g is the acceleration of gravity.

(a) Using the procedure for solving ODEs in terms of Bessel functions from the as in the previous assignment, show that the solution has the form:

$$y(x) = AJ_0\left(\frac{2\omega}{\sqrt{g}}\sqrt{l - x}\right) + BY_0\left(\frac{2\omega}{\sqrt{g}}\sqrt{l - x}\right)$$

It will help to make the transformation $\xi = l - x$.

Engineering Analysis
Problem Set 4

1. Solve the following 1D diffusion problem:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1, \quad t \geq 0$$

subject to the boundary and initial conditions,

$$u(0,t) = 1, \qquad \frac{\partial u(1,t)}{\partial x} + hu(1,t) = 1, \qquad u(x,0) = \sin(\pi x) + x$$

2. Solve the following 1D heat transfer problem:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \qquad 0 < x < L, \quad t \geq 0$$

subject to the boundary and initial conditions,

$$\frac{\partial T(0,t)}{\partial x} = -\frac{H}{k}, \qquad T(L,t) = T_0, \qquad T(x,0) = T_0$$

3. Solve the nonhomogeneous 1-D unsteady diffusion problem defined as follows:

$$c_t = \alpha^2 c_{xx} + Q, \qquad 0 < x < 1$$

$$\frac{\partial c(0,t)}{\partial x} = 0, \qquad c(1,t) = c_0, \qquad c(x,0) = c_0$$

where Q and c_0 are constants. You can save yourself some work by recognizing the similarity of this problem to the previous one.

4. Solve the nonhomogeneous 1-D unsteady diffusion problem defined as follows:

$$u_t = u_{xx} + Ax, \qquad 0 < x < 1$$

$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = f(x) = \frac{A}{6}(x - x^3) + \sin(\pi x)$$

where A is a constant.

5. Heat conduction in a rod where the lateral surface is cooled by convection is governed by the equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - h(T - T_\infty)$$

where h is the convective heat transfer coefficient and T_∞ is the temperature of the surrounding fluid. Show how the change of variables can be used to transform the equation to a simpler form as discussed in class notes. [redacted]. For the specific problem defined below, use separation of variables to solve the problem that results from the transformation and then write out the solution to the original problem.

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - h(T - T_\infty), \qquad 0 < x < L, \quad t \geq 0$$

$$\frac{\partial T(0,t)}{\partial x} = 0, \quad T(L,t) = T_\infty, \quad T(x,0) = T_i$$

Good Luck

Engineering Analysis
Problem Set 5

1. As discussed in class, similarity solutions exist for parabolic PDEs when no physical length scale can be specified. For the following PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right]$$

the solution $u(x,t)$ can be found by first using a similarity transformation to convert the PDE to an ODE. In the above equation, $D(u)$ is a function that depends on the solution variable u. If the similarity variable, $\eta = xt^{\beta}$ in introduced, determine the form of the ODE by using the chain rule to convert the derivatives with respect to x and t to derivatives with respect to η alone. The solution u is now only a function of η, i.e., $u = u(\eta)$. Determine the value of β so that the transformed equation does not contain either x or t explicitly.

2. The 1-D unsteady diffusion problem in a slab subject to a time periodic boundary condition is relevant to a number of application areas.

$$u_t = \kappa u_{xx} \qquad 0 < x < L$$

$$u(0,t) = 0, \qquad u(L,t) = U_0 \cos\omega t \qquad (0 < t < \infty)$$

$$u(x,0) = x$$

Solve the problem by first transforming it to one with homogeneous BCs using the method shown in class (i.e., use superposition and introduce the function that is linear in x but has time-dependent coefficients). This will make the PDE non-homogeneous. Solve the resulting problem using the eigenfunction expansion method covered in class.

3. For a two-dimensional rectangular region in the (x,y) plane, solve the problem for the specified boundary conditions using separation of variables and superposition.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad 0 < x < a, \quad 0 < y < b$$

Boundary conditions:

$$\frac{\partial T}{\partial x}(0,y) = 0$$
$$T(a,y) = T_\infty$$
$$T(x,0) = T_0$$
$$T(x,b) = T_b$$

4. Use the method outlined in class to solve the two-dimensional Poisson equation in a rectangular region with homogeneous Dirichlet boundary conditions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2, \qquad 0 < x < a, \quad 0 < y < b$$

$$u(0,y) = 0$$
$$u(a,y) = 0$$
$$u(x,0) = 0$$
$$u(x,b) = 0$$

Engineering Analysis
Problem Set 6

1. Calculate the solution to the 1-D wave equation for the displacement of a finite string (length, L) for the following boundary and initial conditions:

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < L$$
$$u(0,t) = 0, \qquad u(L,t) = 0$$
$$u(x,0) = f(x), \qquad u_t(x,0) = 0$$

where

$$f(x) = \begin{cases} 0, & 0 < x < 0.25L \\ 4(x/L - 0.25), & 0.25L < x < 0.5L \\ 4(0.75 - x/L), & 0.5L < x < 0.75L \\ 0, & 0.75L < x < L \end{cases}$$

Plot the solution as a mesh plot in the range $0 \leq \tau \leq 2$ and $0 \leq \eta \leq 1$, where $\tau = ct/L$ and $\eta = x/L$.

2. Using separation of variables, solve the 1-D wave equation for the displacement of a finite string (length, L) for the case when the left-end of the string is free to move vertically (i.e., $u_x = 0$ instead of $u = 0$). Here is the complete problem statement:

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < L$$
$$u_x(0,t) = 0, \qquad u(L,t) = 0$$
$$u(x,0) = 0, \qquad u_t(x,0) = (c/10)\cos(\pi x/2L)$$

Write your solution in terms of $x + ct$ and $x - ct$.

3. The 1D wave equation can be used to analyze the longitudinal waves in a rod. Consider a rod of length L, cross-sectional area A, Young's modulus E and mass per unit length ρ. At $x = 0$ the rod is attached to a rigid wall and at $x = L$ the end of the rod is free. A force P is applied to pull the rod at the free end in the axial direction. A new static equilibrium state is attained and the stress in the is uniform and equal to $s_0 = P/A$. If at $t = 0$ the force is removed instantaneously, the subsequent longitudinal displacement of the rod (u) is governed by:

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < L$$
$$u(0,t) = 0, \qquad u_x(L,t) = 0$$
$$u(x,0) = \frac{s_0}{E}x, \qquad u_t(x,0) = 0$$

Use separation of variables to solve for $u(x,t)$. Using this solution, determine the stress at the wall($s(0,t) = Eu_x(0,t)$).

4. Using separation of variables, solve the 1-D damped wave equation for the longitudinal vibrations of a rod in a viscous fluid subject to the conditions below. For this case, the rod is fixed at the left end and is free to oscillate on the right end. Note you can assume that $h < c\pi/(2L)$.

$$u_{tt} + 2hu_t = c^2 u_{xx}, \qquad 0 < x < L$$
$$u(0,t) = 0, \qquad u_x(L,t) = 0$$
$$u(x,0) = x, \qquad u_t(x,0) = 0$$