Chapter 4: Inelastic Material Behaviour

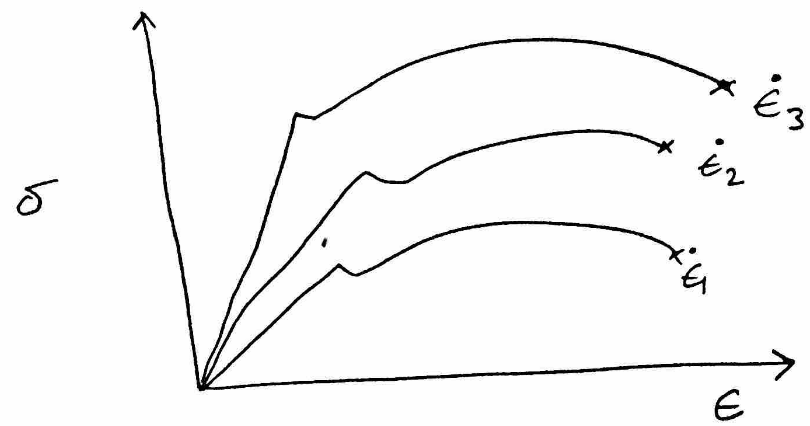
4.1 Limitations on the use of uniaxial stress-strain data

* In tensile, compression and torsion tests we obtain material properties and stress-strain data (E, ν, G) and (σ, ϵ) or (τ, γ).

* In general, those test are executed at certain conditions. such as:
  1. constant temperature (room temperature)
  2. constant rate of loading (constant strain rate $\dot{\epsilon}$)
  3. Load is not being removed.
  4. uniaxial loading condition

* However, this isn't always true in real-life applications so, the limitations of uniaxial stress-strain data are:

  1. Rate of loading (strain rates)
     → Different strain rate results in different material properties and $\sigma-\epsilon$ data



← Different $\dot{\epsilon}$
⇒ Different $\sigma-\epsilon$ curves
⇒ Different mechanical properties.

② Temperature lower than Room Temperature

If a member is loaded at temperature lower than room temperature ⇒ Brittle fracture

③ Temperature Higher than Room Temperature

If Loading with high temperature ⇒ Creep.

④ Unloading & Load reversal

If load is being applied and removed or tension / compression loading ⇒ Fatigue

⑤ Multi axial stress

- For a uniaxial stress → we know the yield stress.
- But, for multiaxial stress → which stress component should be compared to yield stress ?

* In this case, we need to obtain effective uniaxial stress that is function of all stress components.

4.2 Nonlinear material response

After yielding, the material response in $\sigma - \epsilon$ becomes nonlinear and could be classified into 4 types

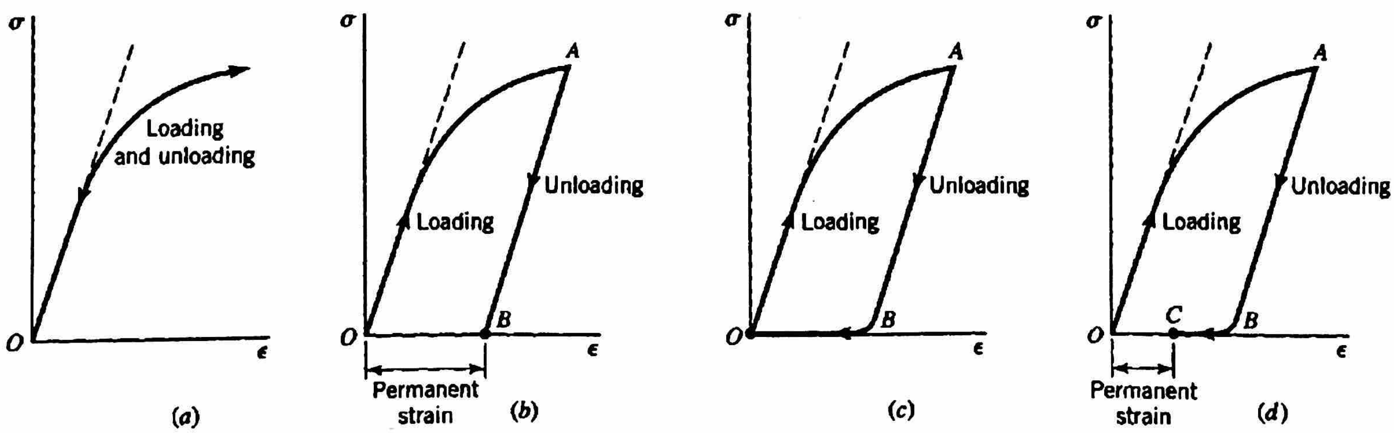
① plastic ② viscoelastic ③ viscoplastic

* For elastic response, after removing the load, the $\sigma - \epsilon$ coincides with loading path and strain goes to zero (Figure a)

* For plastic response, After loading removal, the $\sigma - \epsilon$ does not follow the loading path and strain not zero " permanent strain) (Figure b)

* For viscoelastic response, after loading removal, the $\sigma - \epsilon$ curve doesn't coincide with loading path but strain goes to zero. (Figure c)

* For viscoplastic response, after load removal, the $\sigma - \epsilon$ curve will not coincide with loading path. Deformation recovery will occur but some plastic "permanent" strain will remain (Figure d)



4.3 Yield Criteria: General Concepts and 4.4 Yielding of Ductile materials

* Yield Criteria are applied for multiaxial stress state.

① Maximum principal stress Criterion

If yield stress is (Y) and principal stresses are $\sigma_1 \geq \sigma_2 \geq \sigma_3$

then the yielding occurs when

$$\sigma_1 = Y$$

- this method is usually used for Brittle materials such as concrete

② Maximum principal strain Criterion

* For a material with yield stress $= Y$
  then yield strain ϵ_y is $\epsilon_y = \frac{Y}{E}$, E: Elastic modulus.

* In this method, the yielding occurs when

$$\epsilon_1 = \epsilon_y \quad \text{or} \quad \frac{1}{\cancel{E}}\left(\sigma_1 - \nu(\sigma_2+\sigma_3)\right) = \frac{Y}{\cancel{E}}$$

$$\Rightarrow \boxed{\sigma_1 - \nu(\sigma_2+\sigma_3) = Y}$$

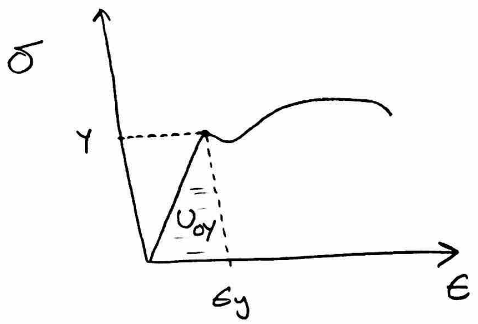
* This is an easy use method as it is normally easier to measure strain.

* Normally used for Brittle materials.

③ Strain-energy Density Criterion

- In this method, the yielding begins when the strain-energy density equals the strain-energy density at Yield in a uniaxial tests.

U_{oy} : Strain-energy Density at yield point



$$U_{oy} = \frac{1}{2} Y \epsilon_y \quad , \quad \epsilon_y = \frac{Y}{E}$$

$$U_{oy} = \frac{1}{2} \frac{Y^2}{E} \qquad \text{— Eq(a)}$$

* the strain-energy density in terms of principal stresses

$$U_o = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right)\right) \qquad \text{— Eq(b)}$$

* then, for yielding to occur

$$U_o = U_{oy} \quad \Rightarrow \quad U_o - U_{oy} = 0$$

Substitute Eq(a) and Eq(b) in Eq(c),

$$\Rightarrow \quad \sigma_e^2 - Y^2 = 0 \quad , \text{ where } \sigma_e = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)}$$

when $\sigma_e = Y \Rightarrow$ yielding

Effective stress.

* This method is further improved for yielding of Ductile materials.

④ Maximum shear-stress Criterion (Tresca Criterion)

From 3D Mohr's Circle

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2}$$

* This method, yielding occurs when

$$\boxed{\tau_{max} = \frac{Y}{2}}$$

* mostly used in Ductile materials.

⑤ Distortional energy-Density Criterion (von Mises Criterion)

strain energy density

$$U_0 = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)\right]$$

Can be written as:-

$$U_0 = U_V + U_D$$

U_V : strain Energy density that causes Volumetric change

U_D : Strain energy density that causes distortion (Shape change)

$$U_0 = \underbrace{\frac{(\sigma_1+\sigma_2+\sigma_3)^2}{K}}_{U_V} + \underbrace{\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2}{12G}}_{U_D}$$

where K and G are Bulk and shear moduli

* In this method, Yielding occurs when $U_D = U_{DY} = \dfrac{Y^2}{6G}$

Distortional SE Density at Yield

* Distortional strain energy density can be written in terms of J_2 "2nd stress invariant of deviatric stress"

as

$$U_D = \frac{1}{2G}\left|J_2\right| , \quad J_2 = -\frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right]$$

For $U_D = U_{DY} = \frac{Y^2}{6G}$ and $U_D = \frac{1}{2G} |J_2|$

$$\Rightarrow \quad \frac{1}{2G} |J_2| = \frac{Y^2}{6G} \Rightarrow |J_2| = \frac{1}{3} Y^2$$

Yield function

$$f = |J_2| - \frac{Y^3}{3} = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right] - \frac{1}{3} Y^2$$

$$\text{Yield} \Rightarrow f=0 \Rightarrow Y^2 = \frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2\right]$$

$\Rightarrow$ Yield $Y = \sigma_e$, σ_e: Effective stress

$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right.} = \sqrt{3|J_2|}$$

$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{xx}-\sigma_{zz}^2) + 3(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2)\right.}$$

σ_e: σ_m $\leftarrow$ Von Mises stress

$\Rightarrow \sigma_e = Y \Rightarrow$ Yielding

or when

$$\tau_{oct} = \frac{\sqrt{2}}{3} Y = 0.471 Y$$

$$J_2 = -\frac{3}{2} \tau_{oct}^2$$

" octahedral shear stress "

4.5 other Yielding Criteria

4.5.1 Mohr-Columb Yield Criterion

* Mostly used for rock and concrete materials.

* For a material with tensile Yield stress is Y_T and compressive yield stress is Y_C. the yield is defined when function f is equal to zero ($f=0$)

$$f = \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi - 2c\cos\phi$$

$f=0 \Rightarrow$ Yielding

where $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are principal stresses

$$c = \frac{Y_T}{2}\sqrt{\frac{Y_C}{Y_T}} = \frac{1}{2}\sqrt{Y_T Y_C}$$

$$\phi = \sin^{-1}\left(\frac{Y_C - Y_T}{Y_C + Y_T}\right)$$

4.5.2 Drucker-Prager Yield Criterion

Yield occurs when $f=0$ (For rock and concrete)

$$f = \alpha I_1 + \sqrt{|J_2|} - K$$

where I_1 is First stress invariant

J_2 is Second invariant of deviatric stress $[\sigma_d]$

Compression $\Leftarrow$ $\alpha = \dfrac{2\sin\phi}{\sqrt{3}(3-\sin\phi)}$, $K = \dfrac{6(c)\cos\phi}{\sqrt{3}(3-\sin\phi)}$

Tension $\Leftarrow$ $\alpha = \dfrac{2\sin\phi}{\sqrt{3}(3+\sin\phi)}$, $K = \dfrac{6(c)\cos\phi}{\sqrt{3}(3+\sin\phi)}$

c and ϕ as in Mohr-Coulumb.

4.5.3 Hill's Criterion for orthotropic materials

Yielding occurs if Yield function f is zero $f = 0$

$$f = F(\sigma_{22} - \sigma_{33})^2 + G(\sigma_{33} - \sigma_{11})^2 + H(\sigma_{11} - \sigma_{22})^2$$
$$+ L(\sigma_{23}^2 + \sigma_{32}^2) + M(\sigma_{13}^2 + \sigma_{31}^2) + N(\sigma_{12}^2 + \sigma_{21}^2) - 1$$

where 1, 2, 3 are x, y, z.

For orthotropic material $\Rightarrow$ 3 Yield Normal stresses X, Y, Z,
and 3 Yield Shear stresses S_{12}, S_{13}, S_{23}

So, the constants F, G, H, L, M and N are

$$2F = \frac{1}{Z^2} + \frac{1}{Y^2} - \frac{1}{X^2}, \quad 2L = \frac{1}{S_{23}^2}$$

$$2G = \frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2}, \quad 2M = \frac{1}{S_{13}^2}$$

$$2H = \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}, \quad 2N = \frac{1}{S_{12}^2}$$

$f = 0 \Rightarrow$ Yield

" All sections and subsection are self-study for deeper discussion — Please re-read from text book "