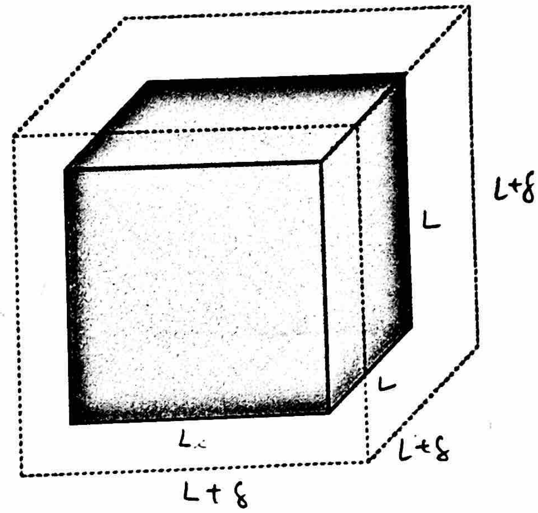
3.4 Equations for thermoelasticity for isotropic materials

Heat $T_1 \rightarrow T_2$
$\Delta T = T_2 - T_1$

For body shown, if ΔT was applied



$\Rightarrow$ Expansion by δ_T " is equal in all directions "

$\delta_T = L\alpha\Delta T$ where α : coefficient of thermal expansion

thermal strain : $\epsilon_T = \dfrac{\delta_T}{L} = \alpha\Delta T$

For mechanical and thermo-mechanical loading of isotropic materials

$\Rightarrow$ Total strain $= \epsilon_{mechanical} + \epsilon_{thermal}$

Thus

$$\epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy}+\sigma_{zz})\right) + \alpha\Delta T$$

$$\epsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu(\sigma_{xx}+\sigma_{zz})\right) + \alpha\Delta T$$

$$\epsilon_{yy} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{xx}+\sigma_{yy})\right) + \alpha\Delta T$$

$$\epsilon_{xy} = \frac{\sigma_{xy}}{2G}, \quad \epsilon_{yz} = \frac{\sigma_{yz}}{2G}, \quad \epsilon_{xz} = \frac{\sigma_{xz}}{2G}$$

$$\Rightarrow \{\epsilon\} = [S]\{\sigma\} \Rightarrow \{\sigma\} = [S]^{-1}\{\epsilon\}$$

$$\sigma_{xx} = \lambda e + 2G\epsilon_{xx} - C\Delta T \qquad e = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$$

$$\sigma_{yy} = \lambda e + 2G\epsilon_{yy} - C\Delta T \qquad C = (3\lambda + 2G)\alpha = \frac{E\alpha}{1-2\nu}$$

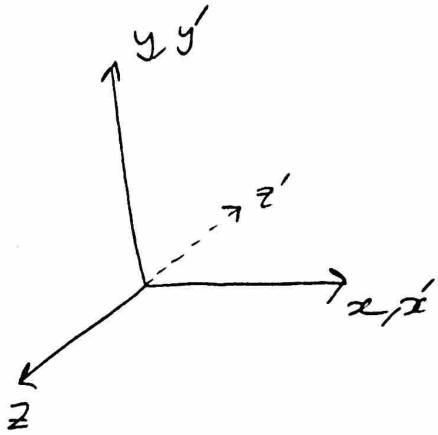
$$\sigma_{zz} = \lambda e + 2G\epsilon_{zz} - C\Delta T$$

$$\sigma_{xy} = 2G\epsilon_{xy}, \quad \sigma_{xz} = 2G\epsilon_{xz}, \quad \sigma_{yz} = 2G\epsilon_{yz}$$

3.5 Hooke's Law: Orthotropic Materials

- Orthotropic materials have 3 orthogonal planes of material symmetry and 3 corresponding orthogonal axes.

- Examples: Wood, reinforced concrete and laminated composites.



* let (x, y, z) original coordinate system
and (x', y', z') new " " .
$(x, y) \rightarrow$ plane of symmetry

Transformation matrix

$$[Q] = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

If $[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$, we can find $[\sigma'] = [Q][\sigma][Q]^T$

$$\Rightarrow \sigma'_{xx} = \sigma_{xx} \; , \; \sigma'_{yy} = \sigma_{yy} \; , \; \sigma'_{zz} = \sigma_{zz} \; , \; \sigma'_{xy} = \sigma_{xy}$$

$$\sigma'_{xz} = -\sigma_{xz} \; , \; \sigma'_{yz} = -\sigma_{yz} \qquad \text{— Eq(1)}$$

and

$$\epsilon'_{xx} = \epsilon_{xx} \; , \; \epsilon'_{yy} = \epsilon_{yy} \; , \; \epsilon'_{zz} = \epsilon_{zz} \; , \; \epsilon'_{xy} = \epsilon_{xy}$$

$$\epsilon'_{xz} = -\epsilon_{xz} \; , \; \epsilon'_{yz} = -\epsilon_{yz} \qquad \text{— Eq(2)}$$

Remember, Hooke's Law for Anisotropic material

$$\sigma_{xx} = C_{11}\,\epsilon_{xx} + C_{12}\,\epsilon_{yy} + C_{13}\,\epsilon_{zz} + 2C_{14}\,\epsilon_{xy} + 2C_{15}\,\epsilon_{xz} + 2C_{16}\,\epsilon_{yz}$$

$$\sigma_{yy} = C_{21}\,\epsilon_{xx} + C_{22}\,\epsilon_{yy} + C_{23}\,\epsilon_{zz} + 2C_{24}\,\epsilon_{xy} + 2C_{25}\,\epsilon_{xz} + 2C_{26}\,\epsilon_{yz}$$

$$\sigma_{zz} = C_{31}\,\epsilon_{xx} + C_{32}\,\epsilon_{yy} + C_{33}\,\epsilon_{zz} + 2C_{34}\,\epsilon_{xy} + 2C_{35}\,\epsilon_{xz} + 2C_{36}\,\epsilon_{yz}$$

$$\sigma_{xy} = C_{41}\,\epsilon_{xx} + C_{42}\,\epsilon_{yy} + C_{43}\,\epsilon_{zz} + 2C_{44}\,\epsilon_{xy} + 2C_{45}\,\epsilon_{xz} + 2C_{46}\,\epsilon_{yz}$$

$$\sigma_{xz} = C_{51}\,\epsilon_{xx} + C_{52}\,\epsilon_{yy} + C_{63}\,\epsilon_{zz} + 2C_{54}\,\epsilon_{xy} + 2C_{55}\,\epsilon_{xz} + 2C_{56}\,\epsilon_{yz}$$

$$\sigma_{yz} = C_{61}\,\epsilon_{xx} + C_{62}\,\epsilon_{yy} + C_{63}\,\epsilon_{zz} + 2C_{64}\,\epsilon_{xy} + 2C_{65}\,\epsilon_{xz} + 2C_{66}\,\epsilon_{yz}$$

For transformed stress state

$$\sigma'_{xx} = C_{11}\,\epsilon'_{xx} + C_{12}\,\epsilon'_{yy} + C_{13}\,\epsilon'_{zz} + 2C_{14}\,\epsilon'_{xy} + 2C_{15}\,\epsilon'_{xz} + 2C_{16}\,\epsilon'_{yz}$$

But from Eq(1), and Eq(2)

$$\sigma_{xx} = \sigma'_{xx} = C_{11}\,\epsilon_{xx} + C_{12}\,\epsilon_{yy} + C_{13}\,\epsilon_{zz} + 2C_{14}\,\epsilon_{xy} - 2C_{15}\,\epsilon_{yz} - 2C_{16}\,\epsilon_{yz}$$

That means $\Rightarrow C_{15} = -C_{15} = 0$ and $C_{16} = -C_{16} = 0$

Do this for all σ_{ij} $\Rightarrow$ $C_{25} = -C_{25} = 0$, $C_{26} = -C_{26} = 0$

$C_{35} = -C_{35} = 0$, $C_{36} = -C_{36} = 0$, $C_{45} = -C_{45} = 0$, $C_{46} = -C_{46} = 0$

$\Rightarrow$ Hooke's Law (matrix form)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\epsilon_{xy} \\ 2\epsilon_{xz} \\ 2\epsilon_{yz} \end{Bmatrix}$$

⇒ For a material with <u>one</u> Plane of symmetry

⇒ we require twelve (12) elastic constants.

* For orthotropic material, we have two additional planes of symmetry (x, z) and (y, z) ⇒ total 3 planes of symmetry.

* In this case, the transformation matrix [Q], becomes

$$[Q] = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

* Following the same procedure, Hooke's Law becomes

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{xz} \\ \epsilon_{yz} \end{Bmatrix}$$

For an orthotropic material (material with 3 planes of symmetry), we need 9 (nine) elastic coefficients.

E_x, E_y, E_z ⇒ Elastic moduli in x, y and z direction.

G_{xy}, G_{xz}, G_{yz} ⇒ Shear moduli in (x, y), (x, z) and (y, z) planes.

$\nu_{xy}, \nu_{xz}, \nu_{yz}$ ⇒ Poisson's ratios in (x, y), (x, z) and (y, z) planes.

Hooke's law, $\sigma = f(\epsilon)$

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix} = \begin{bmatrix} \frac{1-\nu_{yz}\nu_{zy}}{E_y E_z \Delta} & \frac{\nu_{yx}+\nu_{zx}\nu_{yz}}{E_y E_z \Delta} & \frac{\nu_{zx}+\nu_{yx}\nu_{zy}}{E_y E_z \Delta} & 0 & 0 & 0 \\ \frac{\nu_{xy}+\nu_{xz}\nu_{zy}}{E_z E_x \Delta} & \frac{1-\nu_{zx}\nu_{xz}}{E_z E_x \Delta} & \frac{\nu_{zy}+\nu_{zx}\nu_{xy}}{E_z E_x \Delta} & 0 & 0 & 0 \\ \frac{\nu_{xz}+\nu_{xy}\nu_{yz}}{E_x E_y \Delta} & \frac{\nu_{yz}+\nu_{xz}\nu_{yx}}{E_x E_y \Delta} & \frac{1-\nu_{xy}\nu_{yx}}{E_x E_y \Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G_{yz} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G_{zx} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G_{xy} \end{bmatrix} \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix}$$

$$\Delta = \frac{1-\nu_{xy}\nu_{yx}-\nu_{yz}\nu_{zy}-\nu_{zx}\nu_{xz}-2\nu_{xy}\nu_{yz}\nu_{zx}}{E_x E_y E_z}$$

$$\left\{\begin{aligned} \frac{\nu_{yx}+\nu_{zx}\nu_{yz}}{E_y E_z \Delta} &= \frac{\nu_{xy}+\nu_{xz}\nu_{zx}}{E_z E_x \Delta} \\ \frac{\nu_{zy}+\nu_{zx}\nu_{xy}}{E_z E_x \Delta} &= \frac{\nu_{zy}+\nu_{xz}\nu_{yz}}{E_x E_y \Delta} \\ \frac{\nu_{zx}+\nu_{yx}\nu_{zy}}{E_y E_z \Delta} &= \frac{\nu_{xz}+\nu_{xy}\nu_{yz}}{E_x E_y \Delta} \end{aligned}\right\} \text{ To satisfy symmetry}$$

Hooke's law $\epsilon = f(\sigma)$

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{zx}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{xy}} \end{bmatrix} \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix}$$

$$\left.\frac{\nu_{yz}}{E_y} = \frac{\nu_{zy}}{E_z},\ \frac{\nu_{zx}}{E_z} = \frac{\nu_{xz}}{E_x},\ \frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}\right\} \Rightarrow \text{For symmetry}$$

For plane stress $(\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0)$

$$\sigma_{xx} = \frac{E_x}{1 - \nu_{xy}\nu_{yx}} \left(\epsilon_{xx} + \nu_{yx} \epsilon_y \right)$$

$$\sigma_{yy} = \frac{E_y}{1 - \nu_{xy}\nu_{yx}} \left(\epsilon_{yy} + \nu_{xy} \epsilon_{xx} \right)$$

$$\sigma_{xy} = 2G\, \epsilon_{xy}$$

strains

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E_x} - \frac{\nu_{yx}\, \sigma_{yy}}{E_y}$$

$$\epsilon_{yy} = \frac{\sigma_{yy}}{E_y} - \nu_{xy} \frac{\sigma_{xx}}{E_x}$$

$$\epsilon_{zz} = -\left(\nu_{xz} \frac{\sigma_{xx}}{E_x} + \nu_{yz} \frac{\sigma_{yy}}{E_y} \right)$$

$$\epsilon_{xz} = \frac{1}{2G} \sigma_{xy}, \quad \epsilon_{xz} = \epsilon_{yz} = 0$$

plane strain ? Practice Problem.