3.3 Hooke's Law: Isotropic Materials

Strain energy density, Isotropic material

$$U_0 = \frac{1}{2}\lambda(\epsilon_1+\epsilon_2+\epsilon_3)^2 + G(\epsilon_1^2+\epsilon_2^2+\epsilon_3^2)$$

$$\sigma_i = \frac{dU_0}{d\epsilon_i}$$; Principal stress from principal strains:

Thus

$$\sigma_1 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_1+\nu(\epsilon_2+\epsilon_3)\right] = \lambda(\epsilon_1+\epsilon_2+\epsilon_3)+2G\epsilon_1$$

$$\sigma_2 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_2+\nu(\epsilon_1+\epsilon_3)\right] = \lambda(\epsilon_1+\epsilon_2+\epsilon_3)+2G\epsilon_2$$

$$\sigma_3 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_3+\nu(\epsilon_1+\epsilon_2)\right] = \lambda(\epsilon_1+\epsilon_2+\epsilon_3)+2G\epsilon_3$$

* Matrix Form $\{\sigma\} = [C]\{\epsilon\}$
* Then we can find ϵ_i as $\{\epsilon\} = [C]^{-1}\{\sigma\}$

$$\epsilon_1 = \frac{1}{E}(\sigma_1 - \nu(\sigma_2+\sigma_3))$$

$$\epsilon_2 = \frac{1}{E}(\sigma_2 - \nu(\sigma_1+\sigma_3))$$

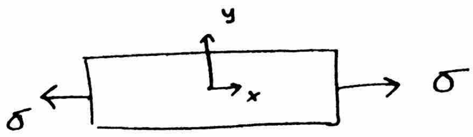
$$\epsilon_3 = \frac{1}{E}(\sigma_3 - \nu(\sigma_1+\sigma_2))$$

* For principal states,

No shear!

Example

For the case of uniaxial tension



$$[\sigma] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using Hooke's Law Equations, Find $[\epsilon]$

Solution

In equations below, $\sigma_{xx} = \sigma$ and all other stress components are zero

$$\epsilon_{xx} = \frac{1}{E}\left(\cancelto{\sigma}{\sigma_{xx}} - \nu\left(\cancelto{0}{\sigma_{yy}} + \cancelto{0}{\sigma_{zz}}\right)\right), \quad \epsilon_{xy} = \frac{1}{2G}\cancelto{0}{\sigma_{xy}}$$

$$\epsilon_{yy} = \frac{1}{E}\left(\cancelto{0}{\sigma_{yy}} - \nu\left(\cancelto{\sigma}{\sigma_{xx}} + \cancelto{0}{\sigma_{zz}}\right)\right), \quad \epsilon_{xz} = \frac{1}{2G}\cancelto{0}{\sigma_{xz}}$$

$$\epsilon_{zz} = \frac{1}{E}\left(\cancelto{0}{\sigma_{zz}} - \nu\left(\cancelto{\sigma}{\sigma_{xx}} + \cancelto{0}{\sigma_{yy}}\right)\right), \quad \epsilon_{yz} = \frac{1}{2G}\cancelto{0}{\sigma_{yz}}$$

$$\Rightarrow \epsilon_{xx} = \frac{\sigma}{E}, \quad \epsilon_{yy} = \frac{-\nu}{E}\sigma, \quad \epsilon_{zz} = \frac{-\nu}{E}\sigma, \quad \epsilon_{xy} = \epsilon_{xz} = \epsilon_{yz} = 0$$

$$[\epsilon] = \begin{bmatrix} \frac{\sigma}{E} & 0 & 0 \\ 0 & \frac{-\nu}{E}\sigma & 0 \\ 0 & 0 & \frac{-\nu}{E}\sigma \end{bmatrix}$$

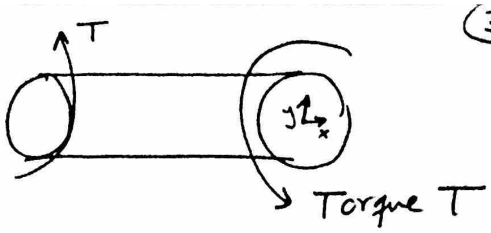
Using tensile tests

$\Rightarrow$ we can obtain two material properties E, ν

- Elastic Modulus (E)
- Poisson's ratio (ν)

Example

For the torsion test, stress state is



$$[\sigma] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tau = \frac{Tr}{J}$$

Find $[\epsilon]$, using Hooke's Law

Solution

In Hooke's law, $\sigma_{xy} = \tau$ all other stress components = zero

$$\Rightarrow \quad \epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = 0$$

$$\epsilon_{xz} = \epsilon_{yz} = 0$$

$$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy} \Rightarrow \epsilon_{xy} = \frac{\tau}{2G}$$

$$[\epsilon] = \begin{bmatrix} 0 & \frac{\tau}{2G} & 0 \\ \frac{\tau}{2G} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using torsion test

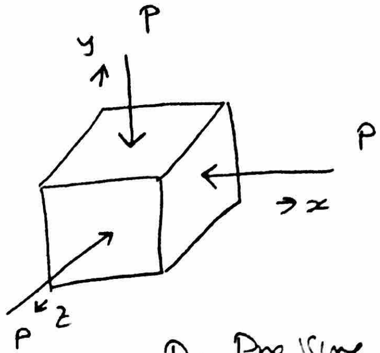
$\Rightarrow$ we can measure one material property $\rightarrow$ G

shear modulus (G).

Example (Problem 3.6)

For the Hydrostatic pressure test shown,

$$[\sigma] = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix}$$



P: Pressure "stress"

Find $[\epsilon]$

In Hooke's law, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -P$

$\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$

$$\epsilon_{xx} = \frac{1}{E}\left(\overset{-P}{\sigma_{xx}} - \nu\left(\overset{-P}{\sigma_{yy}} + \overset{-P}{\sigma_{zz}}\right)\right) \quad \cancel{\epsilon_{xy} = \frac{1}{2G}\sigma_{xy}}\ 0$$

$$\epsilon_{yy} = \frac{1}{E}\left(\overset{-P}{\sigma_{yy}} - \nu\left(\overset{-P}{\sigma_{xx}} + \overset{-P}{\sigma_{zz}}\right)\right) \quad \cancel{\epsilon_{xz} = \frac{1}{2G}\sigma_{xz}}\ 0$$

$$\epsilon_{zz} = \frac{1}{E}\left(\overset{-P}{\sigma_{zz}} - \nu\left(\overset{-P}{\sigma_{xx}} + \overset{-P}{\sigma_{yy}}\right)\right) \quad \cancel{\epsilon_{yz} = \frac{1}{2G}\sigma_{yz}}\ 0$$

$$\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = -\frac{3}{K}P, \quad K = \frac{E}{3(1-2\nu)} \quad \text{Bulk modulus}$$

$$\epsilon_{xy} = \epsilon_{xz} = \epsilon_{yz} = 0$$

From Hydrostatic pressure test

$\Rightarrow$ we can measure $\Rightarrow K$ "one material property"

$$[\epsilon] = \begin{bmatrix} -\frac{3}{K}P & 0 & 0 \\ 0 & \frac{-3}{K}P & 0 \\ 0 & 0 & \frac{-3}{K}P \end{bmatrix}$$

Problem 3.1

$E = 200GPa$, $\nu = 0.29$

Find the principal stress for each point

Principal strain

Strain	Point 1	Point 2	Point 3	Point 4	Point 5
ϵ_1	0.008	0.006	−0.007	0.004	0.009
ϵ_2	−0.002	−0.003	−0.008	−0.005	0.002
ϵ_3	0	0	0	0	0

Solution

$$\sigma_1 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\,\epsilon_1 + \nu(\epsilon_2+\epsilon_3)\right]$$

$$\sigma_2 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\,\epsilon_2 + \nu(\epsilon_1+\epsilon_3)\right]$$

$$\sigma_3 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\,\epsilon_3 + \nu(\epsilon_1+\epsilon_2)\right]$$

$E = 200\times10^9$, $\nu = 0.29$

point 1 $\Rightarrow \epsilon_1 = 0.008,\ \epsilon_2 = -0.002,\ \epsilon_3 = 0$

$\Rightarrow \sigma_1 = 1882.62$ MPa, $\sigma_2 = 332.22$ MPa, $\sigma_3 = 642.3$ MPa

point 2 $\Rightarrow \epsilon_1 = 0.006,\ \epsilon_2 = -0.003,\ \epsilon_3 = 0$

$\Rightarrow \sigma_1 = 1251.39$ MPa, $\sigma_2 = -143.97$ MPa, $\sigma_3 = 321.15$ MPa

point 3 $\Rightarrow \epsilon_1 = -0.007,\ \epsilon_2 = -0.008,\ \epsilon_3 = 0$

$\Rightarrow \sigma_1 = -2691.03$ MPa, $\sigma_2 = -2846.07$ MPa, $\sigma_3 = -1605.75$ MPa

point 4 $\Rightarrow \sigma_1 = 513.11$ MPa, $\sigma_2 = -882.25$, $\sigma_3 = -107.05$ MPa

point 5 $\Rightarrow \sigma_1 = 2572.91$ MPa, $\sigma_2 = 1487.63$ MPa, $\sigma_3 = 1177.55$ MPa

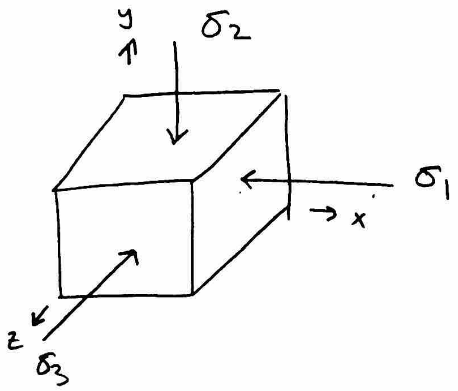
Problem 3.6

For the element and loading shown,

show that

(a) To have no change of volume

$\nu = 0.5$



(b) To have no change of volume while ($\nu \neq 0.5$), the stresses should

$$\sigma_1 + \sigma_2 + \sigma_3 = 0$$

Volume change $\Rightarrow$ Dilatation (Volumetric strain ϵ_v)

$$\epsilon_v = \epsilon_1 + \epsilon_2 + \epsilon_3 = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = I_1$$

(a) For Bulk modulus

$$\sigma_m = K(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) \Rightarrow \sigma_m = K\,\epsilon_v$$

$$\Rightarrow \epsilon_v = \frac{\sigma_m}{K}, \quad \text{If } K = \infty \Rightarrow \epsilon_v = 0 \text{ "No volume change"}$$

$$K = \frac{E}{3(1-2\nu)} \overset{?}{=} \infty, \text{ only if } \boxed{\nu = 0.5 \rightarrow K = \infty} \Rightarrow \boxed{\epsilon_v = 0}$$

(b) $$\sigma_m = K\,\epsilon_v \Rightarrow \epsilon_v = \frac{\sigma_m}{K}, \quad \sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

If $\sigma_m = 0 \rightarrow \epsilon_v = 0$

when $\sigma_m = 0$? only if $\boxed{\sigma_1 + \sigma_2 + \sigma_3 = 0}$

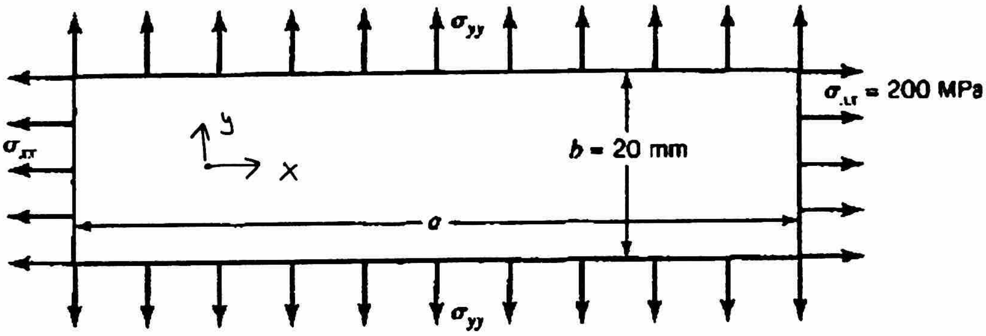
$\Rightarrow$ $\epsilon_v = 0$

Problem 3.15

$E = 72\ GPa\ ,\ \nu = 0.33$

$\sigma_{xx} = 200\ MPa$

1- Find σ_{yy} that make change of length in dimension (a) is zero



2 Find the change of length of dimension (b)

Solution

1 Dimension (a) , $\delta_a = 0 \Rightarrow \epsilon_{xx} = 0$

$$\epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\left(\sigma_{yy} + \cancelto{0}{\sigma_{zz}}\right)\right) \Rightarrow \epsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}\sigma_{yy}$$

$$\epsilon_{xx} = 0 = \frac{\sigma_{xx}}{\cancel{E}} - \frac{\nu}{\cancel{E}}\sigma_{yy} \Rightarrow \sigma_{yy} = \frac{\sigma_{xx}}{0.33}$$

$$\sigma_{yy} = \frac{200\ MPa}{0.33} \Rightarrow \boxed{\sigma_{yy} = 606.06\ MPa}$$

2 $\delta_B = ?$ $\quad \epsilon_{yy} = \frac{\delta_B}{L_B} \Rightarrow \delta_B = \epsilon_{yy} L_B$

$$\epsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}\sigma_{xx} = \frac{606.06 \times 10^6}{72 \times 10^9} - (0.33)\frac{200 \times 10^6}{72 \times 10^9}$$

$$\Rightarrow \epsilon_{yy} = 7.5 \times 10^{-3}$$

$$\Rightarrow \delta_B = (7.5)(10^{-3})(20)(10^{-3})$$

$$\boxed{\delta_B = 150\ \mu m\ \ (150 \times 10^{-6})\ m}$$