3.3 Hooke's Law: Isotropic Materials

* Isotropic materials mechanical properties does not change with direction

* Strain energy density of isotropic material

$$U_0 = \frac{1}{2}\lambda\left(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}\right)^2 + G\left(\epsilon_{xx}^2 + \epsilon_{yy}^2 + \epsilon_{zz}^2 + 2\epsilon_{xy}^2 + 2\epsilon_{xz}^2 + 2\epsilon_{yz}^2\right)$$

or, in terms of principal strains $\epsilon_1, \epsilon_2, \epsilon_3$

$$U_0 = \frac{1}{2}\lambda\left(\epsilon_1 + \epsilon_2 + \epsilon_3\right)^2 + G\left(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2\right)$$

in terms of strain invariants $\bar{I}_1, \bar{I}_2, \bar{I}_3$

$$U_0 = \left(\frac{1}{2}\lambda + G\right)\bar{I}_1^2 - 2G\bar{I}_2$$

* To obtain σ_{ij}, we derive $\frac{dU_0}{d\epsilon_{ij}}$

$$\sigma_{xx} = \lambda\left(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}\right) + 2G\epsilon_{xx}$$

$$\sigma_{yy} = \lambda\left(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}\right) + 2G\epsilon_{yy} \qquad \text{Eq (1)}$$

$$\sigma_{zz} = \lambda\left(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}\right) + 2G\epsilon_{zz}$$

$$\sigma_{xy} = 2G\epsilon_{xy}, \quad \sigma_{xz} = 2G\epsilon_{xz}, \quad \sigma_{yz} = 2G\epsilon_{yz}$$

where $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ Lame's Constant

$G = \frac{E}{2(1+\nu)}$

G: Shear Modulus
E: Elastic modulus
ν: poisson's ratio

In Matrix Form

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} 2G+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2G+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2G+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\epsilon_{xy} \\ 2\epsilon_{xz} \\ 2\epsilon_{yz} \end{Bmatrix}$$

$$\{\sigma\} = [C]\{\epsilon\} \Rightarrow \{\epsilon\} = [C]^{-1}\{\sigma\}$$

Thus,

$$\epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right)$$

$$\epsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right)$$

$$\epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right) \qquad \text{Eq (2)}$$

$$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$

$$\epsilon_{xz} = \frac{1}{2G}\sigma_{xz} = \frac{1+\nu}{E}\sigma_{xz}$$

$$\epsilon_{yz} = \frac{1}{2G}\sigma_{yz} = \frac{1+\nu}{E}\sigma_{yz}$$

Stress invariants as function of Strain invariants

$$I_1 = (3\lambda + 2G)\bar{I}_1 \qquad \text{Eq (3)}$$

$$I_2 = \lambda(3\lambda + 4G)\bar{I}_1^2 + 4G^2\bar{I}_2$$

$$I_3 = \lambda^2(\lambda + 2G)\bar{I}_1^3 + 4\lambda G^2\bar{I}_1\bar{I}_2 + 8G^3\bar{I}_3$$

I_1, I_2, I_3 → stress invariants

$\bar{I}_1, \bar{I}_2$ and $\bar{I}_3$ → strain invariants we can rewrite Eq(1), as:

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\,\epsilon_{xx} + \nu(\epsilon_{yy}+\epsilon_{zz})\right)$$

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\,\epsilon_{yy} + \nu(\epsilon_{xx}+\epsilon_{zz})\right)$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\,\epsilon_{zz} + \nu(\epsilon_{xx}+\epsilon_{yy})\right)$$

$$\sigma_{xy} = \frac{E}{1+\nu}\,\epsilon_{xy}, \quad \sigma_{xz} = \frac{E}{1+\nu}\,\epsilon_{xz}, \quad \sigma_{yz} = \frac{E}{1+\nu}\,\epsilon_{yz}$$

* From equations above, isotropic material has only two "independent" mechanical properties from these two, we can derive all other constants

* Relations between E, ν, λ, G and K

$$E = \frac{G(3\lambda+2G)}{\lambda+G}, \quad \nu = \frac{\lambda}{2(\lambda+G)}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = \frac{3\nu K}{1+\nu}, \quad G = \frac{E}{2(1+\nu)}$$

$$K = \frac{E}{3(1-2\nu)}$$

K bulk modulus $\quad \sigma_M = K(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz})$

mean stress $\sigma_M = \frac{1}{3} I_1, \quad I_1 = \sigma_{xx}+\sigma_{yy}+\sigma_{zz}$

If you remember, for Anisotropic material

the matrix From:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\epsilon_{xy} \\ 2\epsilon_{xz} \\ 2\epsilon_{yz} \end{Bmatrix}$$

In isotropic material

$C_{14} = C_{15} = C_{16} = 0$

$C_{24} = C_{25} = C_{26} = 0$ and by Symmetry

$C_{34} = C_{35} = C_{36} = 0$

$C_{41} + C_{42} + C_{43} = 0$

$C_{51} + C_{52} + C_{53} = 0$

$C_{61} + C_{62} + C_{63} = 0$

Also, $C_{45} = C_{54} = 0$, $C_{46} = C_{64} = 0$, $C_{56} = C_{65} = 0$

But $C_{11} = C_{22} = C_{33} = C_1$

$C_{12} = C_{21} = C_{13} = C_{31} = C_{23} = C_{32} = C_2$

$C_{44} = C_{55} = C_{66} = \dfrac{C_1 - C_2}{2}$

$C_1 = 2G + \lambda$

$C_2 = \lambda$

C_1 and C_2 are the 2 independent material properties

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} C_1 & C_2 & C_2 & 0 & 0 & 0 \\ C_2 & C_1 & C_2 & 0 & 0 & 0 \\ C_2 & C_2 & C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_1 - C_2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_1 - C_2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_1 - C_2}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\epsilon_{xy} \\ 2\epsilon_{xz} \\ 2\epsilon_{yz} \end{Bmatrix}$$

For plane-stress condition $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\epsilon_{xx} + \nu\epsilon_{yy}\right)$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\nu\,\epsilon_{xx} + \epsilon_{yy}\right)$$

$$\{\sigma\} = [C]\{\epsilon\}$$

$$\Rightarrow \{\epsilon\} = [C]^{-1}\{\sigma\}$$

$$\sigma_{xy} = \frac{E}{1+\nu}\,\epsilon_{xy} = 2G\,\epsilon_{xy}$$

$$\Rightarrow \quad \epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\,\sigma_{yy}\right)$$

$$\epsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\,\sigma_{xx}\right)$$

$$\epsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right) = -\frac{\nu}{1-\nu}\left(\epsilon_{xx} + \epsilon_{yy}\right)$$

$$\epsilon_{xy} = \frac{1+\nu}{E}\,\sigma_{xy} = \frac{1}{2G}\,\sigma_{xy}\ , \quad \epsilon_{xz} = \epsilon_{yz} = 0$$

plane strain condition $(\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0)$

$$\sigma_{xx} = \lambda\left(\epsilon_{xx} + \epsilon_{yy}\right) + 2G\,\epsilon_{xx}$$

$$\sigma_{yy} = \lambda\left(\epsilon_{xx} + \epsilon_{yy}\right) + 2G\,\epsilon_{yy}$$

$$\{\sigma\} = [C]\{\epsilon\}$$

$$\Rightarrow \{\epsilon\} = [C]^{-1}\{\sigma\}$$

$$\sigma_{zz} = \lambda\left(\epsilon_{xx} + \epsilon_{yy}\right) = \nu\left(\sigma_{xx} + \sigma_{yy}\right)$$

$$\sigma_{xy} = \frac{E}{1+\nu}\,\epsilon_{xy} = 2G\,\epsilon_{xy}\ , \quad \sigma_{xz} = \sigma_{yz} = 0$$

$$\Rightarrow \quad \epsilon_{xx} = \frac{1+\nu}{E}\left[(1-\nu)\,\sigma_{xx} - \nu\,\sigma_{yy}\right]$$

$$\epsilon_{yy} = \frac{1+\nu}{E}\left[(1-\nu)\,\sigma_{yy} - \nu\,\sigma_{xx}\right]$$

$$\epsilon_{xy} = \frac{1+\nu}{E}\,\sigma_{xy} = \frac{1}{2G}\,\sigma_{xy}$$

In general

① For plane stress $(\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0)$

$\epsilon_{zz} \neq 0$ but ϵ_{xz} and $\epsilon_{yz} = 0$

② For plane strain $(\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0)$

$\sigma_{zz} \neq 0$ but σ_{xz} and $\sigma_{yz} = 0$