Chapter Three: Linear Stress-Strain-Temperature Relations

3.1 First Law of thermodynamics, internal strain energy, and complementary internal strain energy.

First law of thermodynamics:

The work performed on a mechanical system by external forces plus the heat that flows into the system from the outside equals the increase in the internal energy plus the increase in the kinetic energy.

Mathematically,

$$\boxed{\delta W + \delta H = \delta U + \delta K}$$

δW: work done by external forces

δH: Heat flow

δU: Increase in internal energy (potential energy)

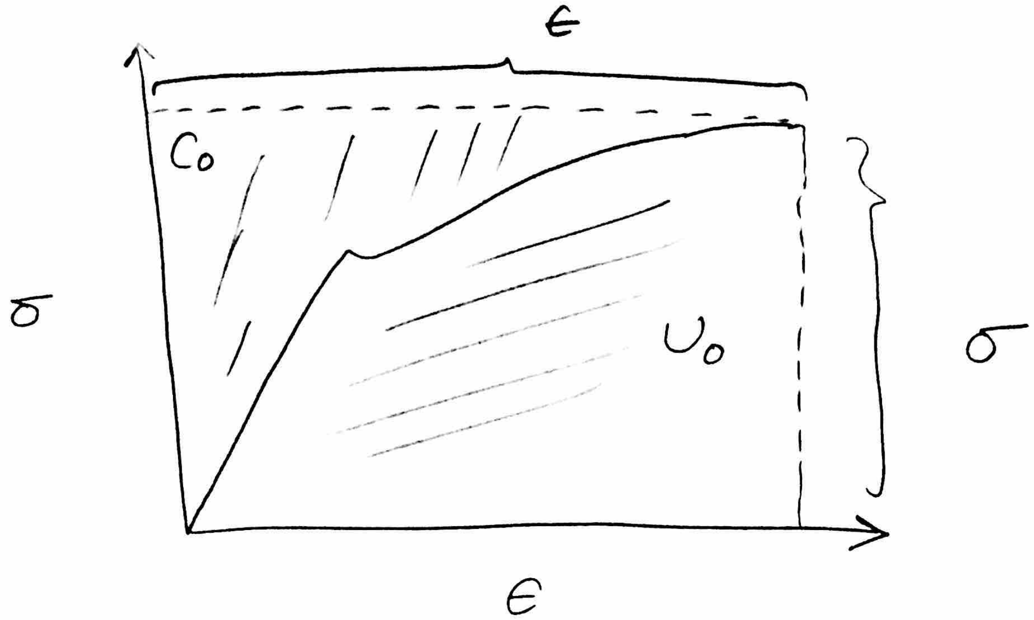
δK: Increase in kinetic energy

* For adiabatic conditions $\delta H = 0$, $\delta K = 0$

No Heat flow $\delta H = 0$
and static equilibrium $\delta K = 0$

$$\Rightarrow \boxed{\delta W = \delta U}$$

For Stress-Strain diagram



U_o is the strain energy density = area under curve

C_o is the complementary strain energy density = area above curve

Total area $= \sigma \epsilon$

$$\Rightarrow \boxed{\sigma \epsilon = C_o + U_o}$$

$$U_o = \int \sigma \, d\epsilon \quad \Rightarrow \quad \sigma = \frac{dU_o}{d\epsilon}$$

$$C_o = \int \epsilon \, d\sigma \quad \Rightarrow \quad \epsilon = \frac{dU_o}{d\sigma}$$

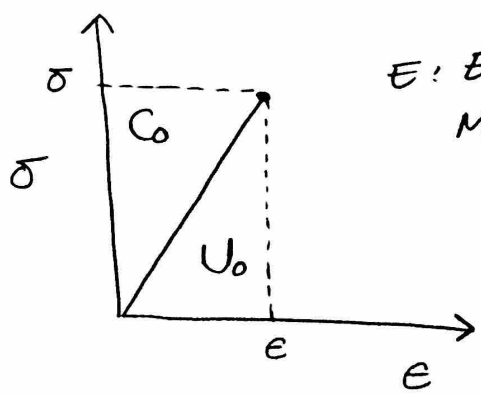
For linear part

$$U_o = \frac{1}{2} \sigma \epsilon , \quad \epsilon = \frac{\sigma}{E}$$

$$U_o = \frac{1}{2} \epsilon^2 E , \quad \frac{dU_o}{d\epsilon} = \sigma = \epsilon E$$

$$C_o = \frac{1}{2} \sigma \epsilon , \quad \sigma = E \epsilon$$

$$C_o = \frac{1}{2} \frac{\sigma^2}{E} , \quad \frac{dU_o}{d\sigma} = \epsilon = \frac{\sigma}{E}$$



E: Elastic Modulus

3.2 Hooke's Law: Anisotropic materials

- For a uniaxial stress state, $\sigma = E\epsilon$ or $\sigma = C\epsilon$, $C = E$

- In general, for an anisotropic material and multiaxial stress state

$$\sigma_{xx} = C_{11}\epsilon_{xx} + C_{12}\epsilon_{yy} + C_{13}\epsilon_{zz} + 2C_{14}\epsilon_{xy} + 2C_{15}\epsilon_{xz} + 2C_{16}\epsilon_{yz}$$

$$\sigma_{yy} = C_{21}\epsilon_{xx} + C_{22}\epsilon_{yy} + C_{23}\epsilon_{zz} + 2C_{24}\epsilon_{xy} + 2C_{25}\epsilon_{xz} + 2C_{26}\epsilon_{yz}$$

$$\sigma_{zz} = C_{31}\epsilon_{xx} + C_{32}\epsilon_{yy} + C_{33}\epsilon_{zz} + 2C_{34}\epsilon_{xy} + 2C_{35}\epsilon_{xz} + 2C_{36}\epsilon_{yz}$$

$$\sigma_{xy} = C_{41}\epsilon_{xx} + C_{42}\epsilon_{yy} + C_{43}\epsilon_{zz} + 2C_{44}\epsilon_{xy} + 2C_{45}\epsilon_{xz} + 2C_{46}\epsilon_{yz}$$

$$\sigma_{xz} = C_{51}\epsilon_{xx} + C_{52}\epsilon_{yy} + C_{53}\epsilon_{zz} + 2C_{54}\epsilon_{xy} + 2C_{55}\epsilon_{xz} + 2C_{56}\epsilon_{yz}$$

$$\sigma_{yz} = C_{61}\epsilon_{xx} + C_{62}\epsilon_{yy} + C_{63}\epsilon_{zz} + 2C_{64}\epsilon_{xy} + 2C_{65}\epsilon_{xz} + 2C_{66}\epsilon_{yz}$$

- where C_{11} to C_{66} are elastic coefficients.

- Anisotropic material requires 36 Elastic coefficients.

Matrix Form

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{xz} \\ \epsilon_{yz} \end{Bmatrix}$$

or,

$$\{\sigma\} = [C]\{\epsilon\}, \quad [C]: \text{ stiffness matrix.}$$

$[C]$ is a symmetric matrix

$$C_{ij} = C_{ji}, \quad C_{12} = C_{21}, \; C_{13} = C_{31} \text{ and so on}$$

Remember $\quad U_0 = \int \sigma \, d\epsilon \quad$ for each σ_{ij} and ϵ_{ij}

Thus,

$$U_0 = \tfrac{1}{2} C_{11} \epsilon_{xx}^2 + \tfrac{1}{2} C_{12} \epsilon_{xx} \epsilon_{yy} + \tfrac{1}{2} C_{13} \epsilon_{xx} \epsilon_{zz} + C_{14} \epsilon_{xx} \epsilon_{xy}$$
$$+ C_{15} \epsilon_{xx} \epsilon_{xz} + C_{16} \epsilon_{xx} \epsilon_{yz}$$
$$+ \tfrac{1}{2} C_{21} \epsilon_{xx} \epsilon_{yy} + \tfrac{1}{2} C_{22} \epsilon_{yy}^2 + \tfrac{1}{2} C_{23} \epsilon_{yy} \epsilon_{zz} + C_{24} \epsilon_{yy} \epsilon_{xy}$$
$$+ C_{25} \epsilon_{yy} \epsilon_{xz} + C_{26} \epsilon_{yy} \epsilon_{yz}$$
$$+ \tfrac{1}{2} C_{31} \epsilon_{xx} \epsilon_{zz} + \tfrac{1}{2} C_{32} \epsilon_{yy} \epsilon_{zz} + \tfrac{1}{2} C_{33} \epsilon_{zz}^2 + C_{34} \epsilon_{zz} \epsilon_{xy}$$
$$+ C_{35} \epsilon_{zz} \epsilon_{xz} + C_{36} \epsilon_{zz} \epsilon_{yz}$$
$$+ C_{41} \epsilon_{xx} \epsilon_{xy} + 2C_{42} \epsilon_{yy} \epsilon_{xy} + C_{43} \epsilon_{zz} \epsilon_{xy} + C_{44} \epsilon_{xy}^2$$
$$+ 2C_{45} \epsilon_{xy} \epsilon_{xz} + 2C_{46} \epsilon_{xy} \epsilon_{yz}$$
$$+ C_{51} \epsilon_{xx} \epsilon_{xz} + C_{52} \epsilon_{yy} \epsilon_{xz} + C_{53} \epsilon_{zz} \epsilon_{xz} + 2C_{54} \epsilon_{xy} \epsilon_{xz}$$
$$+ C_{55} \epsilon_{xz}^2 + 2 C_{56} \epsilon_{xz} \epsilon_{yz}$$
$$+ C_{61} \epsilon_{xx} \epsilon_{yz} + C_{62} \epsilon_{yy} \epsilon_{yz} + C_{63} \epsilon_{zz} \epsilon_{yz} + 2C_{64} \epsilon_{xy} \epsilon_{yz}$$
$$+ 2 C_{65} \epsilon_{xz} \epsilon_{yz} + C_{66} \epsilon_{xz}^2$$

Strain-energy Density for a general Anisotropic Material