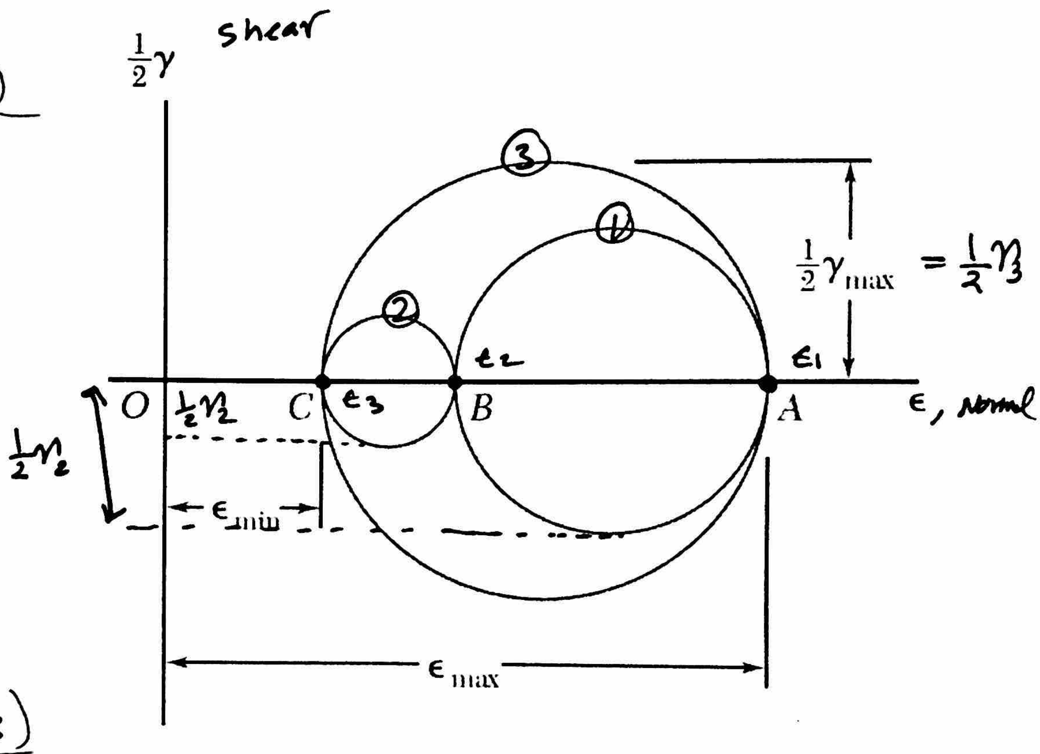
* Mohr's Circle for strain in 3D

* For a strain state
$$[\epsilon] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}$$

* The principal strains are $\epsilon_1 \geq \epsilon_2 \geq \epsilon_3$

- To draw Mohr's Circle, for each pair of principal strains we will have a circle $\Rightarrow (\epsilon_1, \epsilon_2), (\epsilon_2, \epsilon_3), (\epsilon_1, \epsilon_3)$



*** For Circle #1 (ϵ_1, ϵ_2)**

Center $C_1 = \dfrac{\epsilon_1 + \epsilon_2}{2}$

Radius $R_1 = \dfrac{\epsilon_1 - \epsilon_2}{2}$

*** For Circle #2 (ϵ_2, ϵ_3)**

Center $C_2 = \dfrac{\epsilon_2 + \epsilon_3}{2}$

Radius $R_2 = \dfrac{\epsilon_2 - \epsilon_3}{2}$

*** For Circle #3 (ϵ_1, ϵ_3)**

Center $C_3 = \dfrac{\epsilon_1 + \epsilon_3}{2}$

Radius $R_3 = \dfrac{\epsilon_1 - \epsilon_3}{2}$

* Maximum shear strain

$$\frac{1}{2}\gamma_{max} = \frac{\epsilon_1 - \epsilon_3}{2}, \quad \gamma_{max} = \epsilon_1 - \epsilon_3$$

$$\frac{1}{2}\gamma_{max} = \frac{\epsilon_{max} - \epsilon_{min}}{2}. \quad \gamma_{max} = \epsilon_{max} - \epsilon_{min}$$

* Plane strain

* For a 3D strain state $[\epsilon] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}$

* Special case: Plane strain $\Rightarrow \epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$

$$[\epsilon] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_{yy} \end{bmatrix}$$

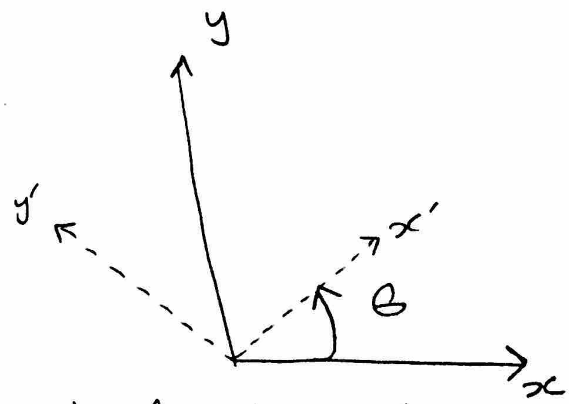
* Compatibility Equation For plane strain

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2 \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$

* Strain Transformation

$[\epsilon']$ is the transformed plane strain state

$$[\epsilon'] = \begin{bmatrix} \epsilon'_{xx} & \epsilon'_{xy} \\ \epsilon'_{xy} & \epsilon'_{yy} \end{bmatrix}$$



$\Rightarrow [\epsilon'] = [Q][\epsilon][Q]^T$ — Eq (1), $[Q]$: transformation matrix

$$[Q] = \begin{bmatrix} \cos(x', x) & \cos(x', y) \\ \cos(y', x) & \cos(y', y) \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(\frac{\pi}{2} - \theta) \\ \cos(\frac{\pi}{2} + \theta) & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

* By evaluating (Eq(1))

$$\epsilon'_{xx} = \epsilon_{xx}\cos^2\theta + \epsilon_{yy}\sin^2\theta + 2\,\epsilon_{xy}\sin\theta\cos\theta \qquad -a$$

$$\epsilon'_{yy} = \epsilon_{xx}\sin^2\theta + \epsilon_{yy}\cos^2\theta - 2\epsilon_{xy}\sin\theta\cos\theta \qquad -b$$

$$\epsilon'_{xy} = -(\epsilon_{xx} - \epsilon_{yy})\sin\theta\cos\theta + \epsilon_{xy}(\cos^2\theta - \sin^2\theta) \qquad -c$$

* If Eq(a) + Eq(b) ⇒ $\epsilon'_{xx} + \epsilon'_{yy} = \epsilon_{xx} + \epsilon_{yy}$ $\qquad -d$

But $\epsilon'_{xx} \neq \epsilon_{xx}$, $\epsilon'_{yy} \neq \epsilon_{yy}$

* Combine Eq(a–d) ⇒ $\epsilon'_{xx}\epsilon'_{yy} - {\epsilon'_{xy}}^2 = \epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}^2$ $\qquad -e$

* For plane strain

strain invariants $\bar{I}_1$, $\bar{I}_2$ and $\bar{I}_3$

$$\bar{I}_1 = \epsilon_{xx} + \epsilon_{yy}$$

$$\bar{I}_2 = \epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}^2$$

$$\bar{I}_3 = 0$$

* Mohr's Circle for Strain in 2D

we can re-write Eq(a) – Eq(b) – Eq(c), as:-

$$\epsilon'_{xx} = \frac{1}{2}(\epsilon_{xx} + \epsilon_{yy}) + \frac{1}{2}(\epsilon_{xx} - \epsilon_{yy})\cos 2\theta + \epsilon_{xy}\sin 2\theta \quad -(A)$$

$$\epsilon'_{yy} = \frac{1}{2}(\epsilon_{xx} + \epsilon_{yy}) - \frac{1}{2}(\epsilon_{xx} - \epsilon_{yy})\cos 2\theta - \epsilon_{xy}\sin 2\theta \quad -(B)$$

$$\epsilon'_{xy} = -\frac{1}{2}(\epsilon_{xx} - \epsilon_{yy})\sin 2\theta + \epsilon_{xy}\cos 2\theta \quad -(C)$$

Re-write Eq(A) and Eq(C), let $\epsilon_{Avg} = \frac{1}{2}(\epsilon_{xx} + \epsilon_{yy})$

$$(\epsilon'_{xx} - \epsilon_{avg})^2 = \Big(\underbrace{\frac{1}{2}(\epsilon_{xx} - \epsilon_{yy})}_{A}\cos 2\theta + \underbrace{\epsilon_{xy}}_{B}\sin 2\theta\Big)^2 \quad +$$

$$(\epsilon'_{xy})^2 = \Big(-\underbrace{\frac{1}{2}(\epsilon_{xx} - \epsilon_{yy})}_{A}\sin 2\theta + \underbrace{\epsilon_{xy}}_{B}\cos 2\theta\Big)^2$$

$$(\epsilon'_{xx} - \epsilon_{Avg})^2 + (\epsilon'_{xy})^2 = \frac{1}{4}(\epsilon_{xx} - \epsilon_{yy})^2 + \epsilon_{xy}^2$$

This is an equation of circle "Mohr's Circle for 2D Strain"

$$(A\cos 2\theta + B\sin 2\theta)^2 = A^2\cos^2 2\theta + \cancel{2AB\sin 2\theta\cos 2\theta} + B^2\sin^2 2\theta$$

$$(-A\sin 2\theta + B\cos 2\theta)^2 = A^2\sin^2 2\theta - \cancel{2AB\sin 2\theta\cos 2\theta} + B^2\cos^2 2\theta \quad +$$

$$A^2(\underbrace{\sin^2 2\theta + \cos^2 2\theta}_{=1}) + B^2(\underbrace{\sin^2 2\theta + \cos^2 2\theta}_{=1})$$

$$\Rightarrow A^2 + B^2$$

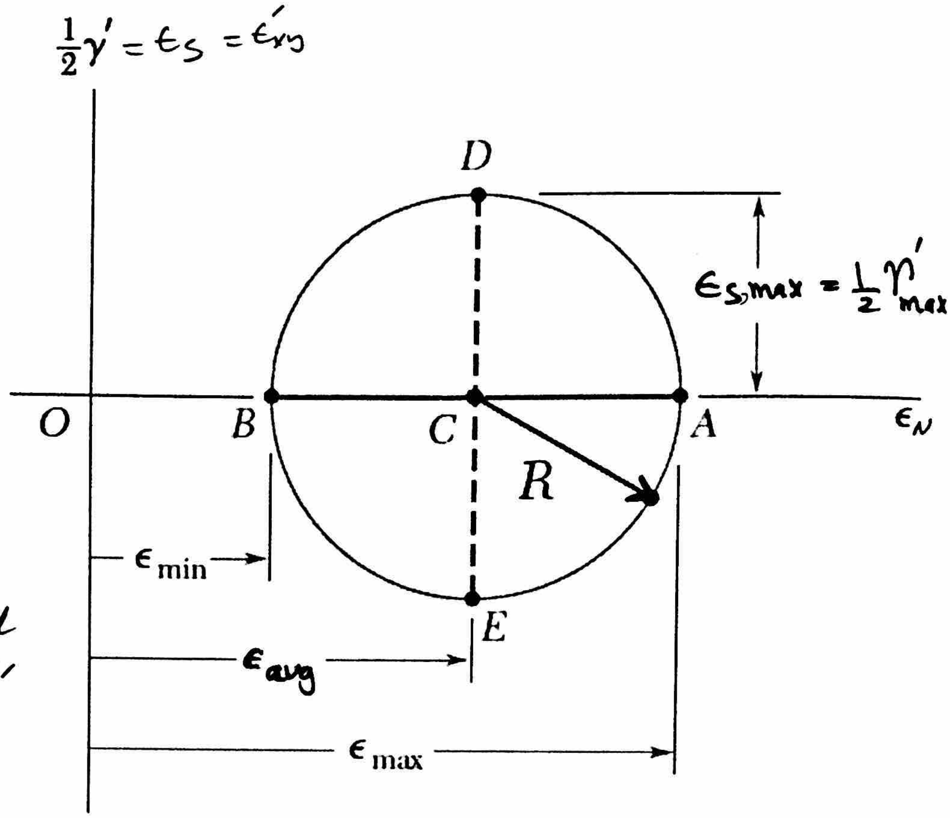
$$\Rightarrow \frac{1}{4}(\epsilon_{xx} - \epsilon_{yy})^2 + \epsilon_{xy}^2$$

The center of this circle (C)

$$C = \epsilon_{avg} = \frac{\epsilon_{xx} + \epsilon_{yy}}{2}$$

* The Radius is (R)

$$R = \sqrt{\frac{1}{4}(\epsilon_{xx} - \epsilon_{yy})^2 + \epsilon_{xy}^2}$$



- points A and B

- Point A

$$\epsilon_1 = \epsilon_{Avg} + R$$ "Max principal strain"

- point B

$$\epsilon_2 = \epsilon_{Avg} - R$$ "min principal strain"

- points D or E

$$\epsilon_s = \epsilon_{s,max} = R$$

or $\gamma'_{max} = 2R = \epsilon_{max} - \epsilon_{min}$

Question ① what is θ that makes $\epsilon_s = \epsilon'_{xy} = 0$? θ_p

ANSWER ⇒ Eq(C) ⇒ $\epsilon'_{xy} = -\frac{1}{2}(\epsilon_{xx} - \epsilon_{yy})\sin 2\theta + \epsilon_{xy}\cos 2\theta$

let $= 0$ ⇒ $\tan 2\theta_p = \dfrac{2\epsilon_{xy}}{\epsilon_{xx} - \epsilon_{yy}}$

principal strain plane

NO shear strain

$\epsilon'_{xy} = 0$

Question ② ⇒ what is θ that make max shear $\epsilon'_{xy} = \epsilon_{s,max}$?

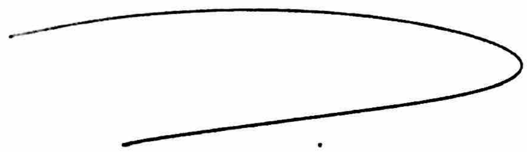
ANSWER ⇒ Eq(A) ⇒ $\epsilon'_{xx} = \epsilon_{Avg} + \frac{1}{2}(\epsilon_{xx} - \epsilon_{yy})\cos 2\theta + \epsilon_{xy}\sin 2\theta$

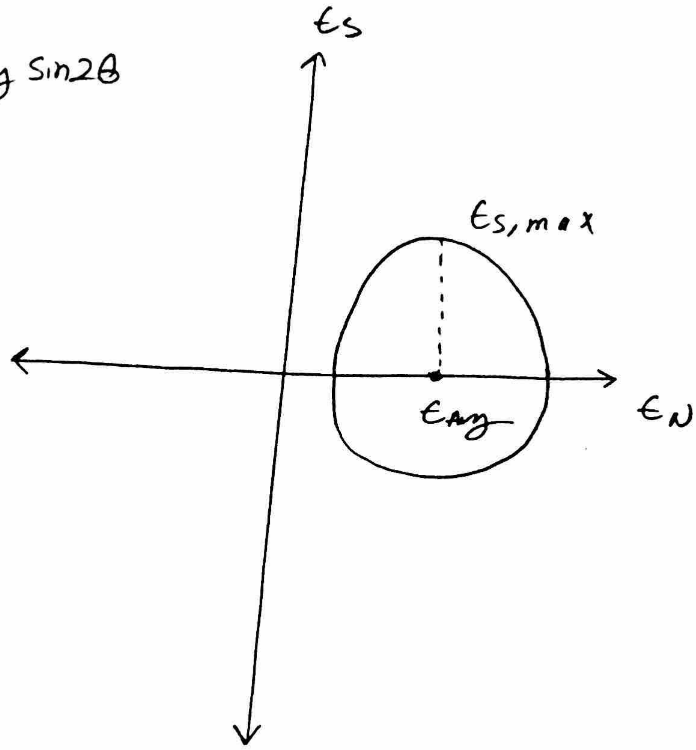
$\epsilon_{s,max}$ happens when $\epsilon'_{xx} = \epsilon_{Avg}$

So, $\epsilon_{avg} = \epsilon_{Avg} + \frac{1}{2}(\epsilon_{xx} - \epsilon_{yy})\cos 2\theta + \epsilon_{xy}\sin 2\theta$

$$\Rightarrow \tan 2\theta_s = -\frac{\epsilon_{xx} - \epsilon_{yy}}{2\epsilon_{xy}}$$

planes of max. shear strain





Application on plane strain transformation

⇒ Strain measurement using strain gages and strain Rosettes.