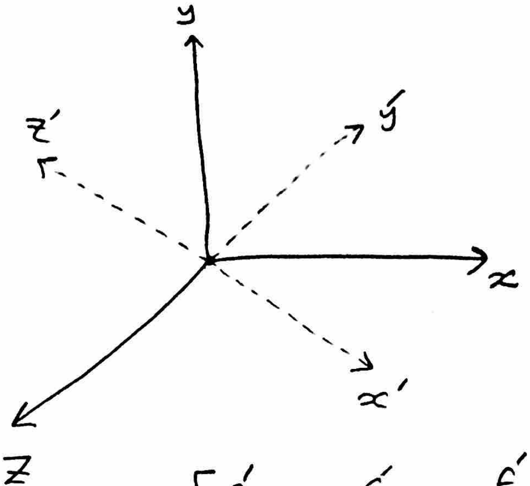
2.7.4 Strain transformations

$$[\epsilon] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}$$



To find the transformed strain state $[\epsilon']$

$$[\epsilon'] = [Q][\epsilon][Q^T] \;, \qquad [\epsilon'] = \begin{bmatrix} \epsilon'_{xx} & \epsilon'_{xy} & \epsilon'_{xz} \\ \epsilon'_{xy} & \epsilon'_{yy} & \epsilon'_{yz} \\ \epsilon'_{xz} & \epsilon'_{yz} & \epsilon'_{zz} \end{bmatrix}$$

$[Q]$ is the transformation matrix

$$[Q] = \begin{bmatrix} \cos(x',x) & \cos(x',y) & \cos(x',z) \\ \cos(y',x) & \cos(y',y) & \cos(y',z) \\ \cos(z',x) & \cos(z',y) & \cos(z',z) \end{bmatrix} , \quad [Q] = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix}$$

* Some properties of $[Q]$

① $[Q]^{-1} = [Q]^T \Rightarrow [Q]^{-1}[Q] = [Q][Q]^T = [I]$

②
$$\left.\begin{aligned} \ell_1^2 + \ell_2^2 + \ell_3^2 &= 1 \\ m_1^2 + m_2^2 + m_3^3 &= 1 \\ n_1^2 + n_2^2 + n_3^2 &= 1 \end{aligned}\right\} \quad \begin{aligned} \ell_1\ell_2 + m_1m_2 + n_1n_2 &= 0 \\ \ell_1\ell_3 + m_1m_3 + n_1n_3 &= 0 \\ \ell_2\ell_3 + m_2m_3 + n_2n_3 &= 0 \end{aligned} \quad \left\{\begin{aligned} &\ell_1^2 + m_1^2 + n_1^2 = 1 \\ &\ell_2^2 + m_2^2 + n_2^2 = 1 \\ &\ell_3^2 + m_3^3 + n_3^2 = 1 \\ &\Rightarrow \ell_i^2 + m_i^3 + n_i^2 = 1 \\ &(i = 1,2,3) \end{aligned}\right.$$

③
$$\ell_1 m_1 + \ell_2 m_2 + \ell_3 m_3 = 0 \;, \quad \sum_{i=1}^{3} \ell_i m_i = 0$$

$$\ell_1 n_1 + \ell_2 n_2 + \ell_3 n_3 = 0 \;, \quad \sum_{i=1}^{3} \ell_i n_i = 0$$

$$m_1 n_1 + m_2 n_2 + m_3 n_3 = 0 \;, \quad \sum_{i=1}^{3} m_i n_i = 0$$

From $[\epsilon'] = [Q][\epsilon][Q]^T$ ②

$$\epsilon'_{xx} = l_1^2 \epsilon_{xx} + m_1^2 \epsilon_{yy} + n_1^2 \epsilon_{zz} + 2 l_1 m_1 \epsilon_{xy} + 2 l_1 n_1 \epsilon_{xz} + 2 m_1 n_1 \epsilon_{yz}$$

$$\epsilon'_{yy} = l_2^2 \epsilon_{xx} + m_2^2 \epsilon_{yy} + n_2^2 \epsilon_{zz} + 2 l_2 m_3 \epsilon_{xy} + 2 l_2 n_2 \epsilon_{xz} + 2 m_2 n_2 \epsilon_{yz}$$

$$\epsilon'_{zz} = l_3^2 \epsilon_{xx} + m_3^2 \epsilon_{yy} + n_3^2 \epsilon_{zz} + 2 l_3 m_3 \epsilon_{xy} + 2 l_3 n_3 \epsilon_{xz} + 2 m_3 n_3 \epsilon_{yz}$$

$$\epsilon'_{xy} = \frac{1}{2}\gamma'_{xy} = l_1 l_2 \epsilon_{xx} + m_1 m_2 \epsilon_{yy} + n_1 n_2 \epsilon_{zz} + (l_1 m_2 + l_2 m_1) \epsilon_{xy}$$

$$+ (m_1 n_2 + m_2 n_1) \epsilon_{yz} + (l_1 n_2 + l_2 n_1) \epsilon_{xz}$$

$$\epsilon'_{yz} = \frac{1}{2}\gamma'_{yz} = l_2 l_3 \epsilon_{xx} + m_2 m_3 \epsilon_{yy} + n_2 n_3 \epsilon_{zz} + (l_2 m_3 + l_3 m_2) \epsilon_{xy}$$

$$+ (m_2 n_3 + m_3 n_2) \epsilon_{yz} + (l_2 n_3 + l_3 n_2) \epsilon_{xz}$$

$$\epsilon'_{xz} = \frac{1}{2}\gamma'_{xz} = l_1 l_3 \epsilon_{xx} + m_1 m_3 \epsilon_{yy} + n_1 n_3 \epsilon_{zz} + (l_1 m_3 + l_3 m_1) \epsilon_{xy}$$

$$+ (m_1 n_3 + m_3 n_1) \epsilon_{yz} + (l_1 n_3 + l_3 n_1) \epsilon_{xz}$$

Just like stresses.

2.7.4

* Principal strains and principal Directions

For a strain state $[\epsilon]$, we can find the principal values ϵ_i $(i=1,2,3)$ and associated principal directions $\{S_i\}$ $(i=1,2,3)$ as:-

$$[\epsilon]\{S_i\} = \epsilon_i \{S_i\} \quad , \quad [\epsilon] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}$$

To find ϵ_i and $\{S_i\}$

$$[\epsilon]\{S_i\} - \epsilon_i\{S_i\} = 0 \Rightarrow \underbrace{\left([\epsilon] - \epsilon_i[I]\right)}_{\det(\,)=0} \underbrace{\{S_i\}}_{\neq 0} = 0$$

$$\left([\epsilon] - \epsilon_i[I]\right) = \begin{bmatrix} \epsilon_{xx}-\epsilon_i & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy}-\epsilon_i & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz}-\epsilon_i \end{bmatrix}$$

$$\det(\,) = 0 \Rightarrow \epsilon_i^3 - \bar{I}_1 \epsilon_i^2 + \bar{I}_2 \epsilon_i - \bar{I}_3 = 0$$

principal Equation

↳ principal strain $\epsilon_1 \geq \epsilon_2 \geq \epsilon_3$

$\bar{I}_1$, $\bar{I}_2$ and $\bar{I}_3$ are strain invariants

$$\bar{I}_1 = tr(\epsilon) = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$$

$$\bar{I}_2 = \epsilon_{xx}\epsilon_{yy} + \epsilon_{xx}\epsilon_{zz} + \epsilon_{yy}\epsilon_{zz} - \epsilon_{xy}^2 - \epsilon_{xz}^2 - \epsilon_{yz}^2$$

$$\bar{I}_2 = \frac{1}{2}\left[tr^2(\epsilon) - tr(\epsilon^2)\right], \quad [\epsilon]^2 = [\epsilon][\epsilon]$$

$$\bar{I}_3 = \det([\epsilon])$$

* Then we can obtain principal strain directions $\{S_1\}$, $\{S_2\}$ and $\{S_3\}$

* Then the normalized directions can be obtained $\{S_{n1}\}$, $\{S_{n2}\}$ and $\{S_{n3}\}$ as we learned previously.

Note

$$[Q] = \left[\{S_{n1}\}^T ; \{S_{n2}\}^T ; \{S_{n3}\} \right]$$

row-by-row

transformation matrix

$$[\epsilon'] = [Q][\epsilon][Q]^T = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}$$

For $[\epsilon']$

$$\bar{I}_1 = \epsilon_1 + \epsilon_2 + \epsilon_3$$

$$\bar{I}_2 = \epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3$$

$$\bar{I}_3 = \epsilon_1\epsilon_2\epsilon_3$$

* Mean & Deviatric strains

* For a strain state (or field) called $[\epsilon]$

$$[\epsilon] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}$$

* The mean strain (ϵ_M) is defined as:

$$\epsilon_M = \frac{1}{3} \bar{I}_1 = \frac{1}{3}(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) = \frac{1}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3)$$

* Then the mean strain matrix $[\epsilon_m]$ is

$$[\epsilon_m] = \begin{bmatrix} \epsilon_M & 0 & 0 \\ 0 & \epsilon_M & 0 \\ 0 & 0 & \epsilon_M \end{bmatrix}$$

* Define the deviatric strain matrix $[\epsilon_d]$ as:

$$[\epsilon_d] = [\epsilon] - [\epsilon_m]$$

$$\Rightarrow \quad [\epsilon_d] = \begin{bmatrix} \epsilon_{xx} - \epsilon_M & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} - \epsilon_M & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} - \epsilon_M \end{bmatrix}$$

Mean & deviatric strains are Important in failure theories "Chapter 4"

· For $[\epsilon_d] = \begin{bmatrix} \epsilon_{xx} - \epsilon_M & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} - \epsilon_M & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} - \epsilon_M \end{bmatrix}$

we can find the principal values (ϵ_{di}, $i=1,2,3$) and principal Directions $\{S_{di}\}$ $(i=1,2,3)$, following the same procedure. Thus

$$\epsilon_d^3 - \bar{J}_1 \epsilon_d^2 + \bar{J}_2 \epsilon_d - \bar{J}_3 = 0 \Rightarrow$$ principal values of $[\epsilon_d]$ $\epsilon_{d1} \geq \epsilon_{d2} \geq \epsilon_{d3}$

$\bar{J}_1$, $\bar{J}_2$ and $\bar{J}_3$ are the deviatoric strain invariants:

$$\bar{J}_1 = tr(\epsilon_d) = 0$$

$$\bar{J}_2 = \frac{1}{2}\left[tr^2(\epsilon_d) - tr(\epsilon_d^2)\right] \quad , \quad [\epsilon_d]^2 = [\epsilon_d][\epsilon_d]$$

$$= -\frac{1}{6}\left[(\epsilon_{xx} - \epsilon_{yy})^2 + (\epsilon_{xx} - \epsilon_{zz})^2 + (\epsilon_{yy} - \epsilon_{zz})^2 + 6(\epsilon_{xy}^2 + \epsilon_{xz}^2 + \epsilon_{yz}^2)\right]$$

$$= -\frac{1}{6}\left[(\epsilon_1 - \epsilon_2)^2 + (\epsilon_1 - \epsilon_3)^2 + (\epsilon_2 - \epsilon_3)^2\right]$$

$$= \bar{I}_2 - \frac{1}{3}\bar{I}_1^2$$

$$\bar{J}_3 = det(\epsilon_d) = (\epsilon_1 - \epsilon_M)(\epsilon_2 - \epsilon_M)(\epsilon_3 - \epsilon_M)$$

$$= \bar{I}_3 - \frac{1}{3}\bar{I}_1\bar{I}_2 + \frac{2}{27}\bar{I}_1^3$$

· thus, the principal values of $[\epsilon d]$, are

① $\epsilon_{d_1} = \epsilon_1 - \epsilon_M = \dfrac{(\epsilon_1-\epsilon_3)+(\epsilon_1-\epsilon_2)}{3}$

② $\epsilon_{d_2} = \epsilon_2 - \epsilon_M = \dfrac{(\epsilon_2-\epsilon_3)+(\epsilon_2-\epsilon_1)}{3}$

③ $\epsilon_{d_3} = \epsilon_3 - \epsilon_M = \dfrac{(\epsilon_3-\epsilon_1)+(\epsilon_3-\epsilon_2)}{3}$

$\Rightarrow \epsilon_{d_i} = \epsilon_i - \epsilon_M$

From above equations $\Rightarrow \epsilon_{d_1} + \epsilon_{d_2} + \epsilon_{d_3} = 0$

* For principal Directions $\{S_{d_i}\}$

$\{S_{d_i}\} = \{S_i\}$, normalized $\{S_{dn_i}\} = \{S_{n_i}\}$

" The principal directions of $[\epsilon d]$ are the same as those of $[\epsilon]$ "

* Octahedral strains

For an octahedral plane with $\vec{N} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

The engineering octahedral shear strain (γ_{oct}), is:

$$\gamma_{oct} = \frac{2}{3}\sqrt{(\epsilon_1 - \epsilon_2)^2 + (\epsilon_1 - \epsilon_3)^2 + (\epsilon_2 - \epsilon_3)^2}$$

$$= \frac{2}{3}\sqrt{(\epsilon_{xx} - \epsilon_{yy})^2 + (\epsilon_{xx} - \epsilon_{zz})^2 + (\epsilon_{yy} - \epsilon_{zz})^2 + 6(\epsilon_{xy}^2 + \epsilon_{xz}^2 + \epsilon_{yz}^2)}$$

$$= 2\sqrt{\frac{2}{9}\bar{I}_1^2 - \frac{2}{3}\bar{I}_2}$$

Where, the octahedral shear strain (ϵ_{oct}^{s}), is

$$\boxed{\epsilon_{oct}^{s} = \frac{1}{2}\gamma_{oct}}$$

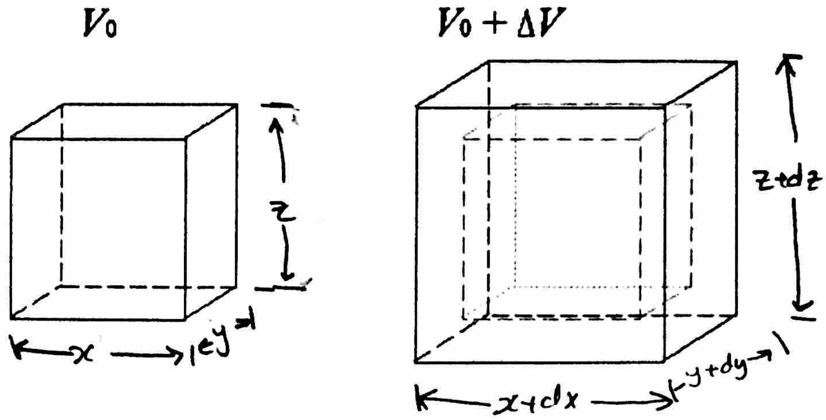
The octahedral normal strain (ϵ_{oct}^{N}), is

$$\epsilon_{oct}^{N} = \frac{1}{3}\bar{I}_1 = \frac{1}{3}(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})$$

$$= \frac{1}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3)$$

* Volumetric strain (ϵ_v) - Dilatation

Dilatation is the relative change of the volume



$$\epsilon_v = \frac{\Delta V}{V_0} = \frac{V_{new} - V_0}{V_0}$$

$V_0 = (x)(y)(z)$ — Divide by $xyz \Rightarrow V_0 = 1$

$V_{new} = (x+dx)(y+dy)(z+dz) \Rightarrow$ divide by $xyz \Rightarrow$

$\epsilon = \frac{\delta}{L}$

$$= (1+\epsilon_{xx})(1+\epsilon_{yy})(1+\epsilon_{zz})$$

$$\Rightarrow \quad \epsilon_v = \frac{(1+\epsilon_{xx})(1+\epsilon_{yy})(1+\epsilon_{zz}) - 1}{1}$$

$$= 1 + \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} + \cancel{\epsilon_{xx}\epsilon_{yy}} + \cancel{\epsilon_{xx}\epsilon_{zz}} + \cancel{\epsilon_{yy}\epsilon_{zz}} + \cancel{\epsilon_{xx}\epsilon_{yy}\epsilon_{zz}} - 1$$

small displacement theory

$$\Rightarrow \quad \epsilon_v = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$$

$$= \epsilon_1 + \epsilon_2 + \epsilon_3$$

$$= I_1$$