2.8 Small-Displacement theory

For a displacement vector $\vec{r} = u\hat{i} + v\hat{j} + w\hat{k}$

$u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$

Normal strain

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\epsilon_{xy} = \epsilon_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \gamma_{xy} = \epsilon_{xy} + \epsilon_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\epsilon_{xz} = \epsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \quad \gamma_{xz} = \epsilon_{xz} + \epsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$e_{yz} = \epsilon_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad \gamma_{yz} = \epsilon_{yz} + \epsilon_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

shear strain

γ_{ij} is the engineering shear strain

ϵ_{ij} is the shear strain component in $[\epsilon]$ matrix.

Example

Displacement vector $\vec{r} = u\hat{i} + v\hat{j} + w\hat{k}$

$u = Ax^2y$, $v = Byz$, $w = Cxz^3$

Find $[\epsilon]$

Solution

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = 2Axy, \quad \epsilon_{yy} = \frac{\partial v}{\partial y} = Bz, \quad \epsilon_{zz} = \frac{\partial w}{\partial z} = 3Cxz^2$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(Ax^2 + 0\right) = \frac{Ax^2}{2}$$

$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\left(By + 0\right) = \frac{By}{2}$$

$$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}\left(Cz^3 + 0\right) = \frac{Cz^3}{2}$$

$$[\epsilon] = \begin{bmatrix} 2Axy & \frac{Ax^2}{2} & \frac{Cz^3}{2} \\ \frac{Ax^2}{2} & Bz & \frac{By}{2} \\ \frac{Cz^3}{2} & \frac{By}{2} & 3Cxz^2 \end{bmatrix}$$

2.8.1 Strain Compatibility equations

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z}$$

$$2\epsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad 2\epsilon_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad 2\epsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

* let's Find

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2}$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y}$$

and compare to

$$2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} = \underbrace{\frac{\partial^3 u}{\partial x \partial y^2}}_{\frac{\partial^2 \epsilon_{xx}}{\partial y^2}} + \underbrace{\frac{\partial^3 v}{\partial x^2 \partial y}}_{\frac{\partial^2 \epsilon_{yy}}{\partial x \partial y}}$$

$$\Rightarrow \quad \frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$

* Doing the same for all other strain components

$\Rightarrow$

$$\frac{\partial^2 \epsilon_{yy}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial y^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \epsilon_{xz}}{\partial x \partial z}$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial z^2} = 2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} + \frac{\partial^2 \epsilon_{xy}}{\partial z^2} = \frac{\partial^2 \epsilon_{yz}}{\partial z \partial x} + \frac{\partial^2 \epsilon_{zx}}{\partial y \partial z}$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial x \partial z} + \frac{\partial^2 \epsilon_{xz}}{\partial y^2} = \frac{\partial^2 \epsilon_{xy}}{\partial y \partial z} + \frac{\partial^2 \epsilon_{yz}}{\partial x \partial y}$$

$$\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} + \frac{\partial^2 \epsilon_{yz}}{\partial x^2} = \frac{\partial^2 \epsilon_{xz}}{\partial x \partial y} + \frac{\partial^2 \epsilon_{xy}}{\partial x \partial z}$$

Eq (2.83) text book

* These Equations (6 equations) are called "strain compatibility equations"

* this means that for strain components ($\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz}$) In small-displacement theory are possible strain component if and only if they satisfy these six Equations.

* They are available in Polar and cylindrical coordinates Eq(2.85) – Eq(2.87)

Strain Compatibility Equations can be written in Form of 3 equation of 4^{th} order as:-

$$\frac{\partial^4 e_{xx}}{\partial y^2 \partial z^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right)$$

$$\frac{\partial^4 e_{yy}}{\partial z^2 \partial x^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x}\right)$$

$$\frac{\partial^4 e_{zz}}{\partial x^2 \partial y^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y}\right)$$

where e_{ij} is the strain $i = x, y, z$ $j = x, y, z$

Example: For the following strain field,

$$\epsilon_{xx} = Ay^3 \ , \ \epsilon_{yy} = Ax^3 \ , \ \epsilon_{xy} = Bxy(x+y)$$

$$\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$$

and A and B are constants, where $A = +\frac{2B}{3}$

show that this strain field is valid and satisfies the strain Compatibility Equations.

solution

using $$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$

$$\frac{\partial \epsilon_{xx}}{\partial y} = 3Ay^2 \Rightarrow \frac{\partial^2 \epsilon_{xx}}{\partial y^2} = 6Ay = (6)\left(\tfrac{2}{3}B\right)y = 4By$$

$$\frac{\partial \epsilon_{yy}}{\partial x} = 3Ax^2 \Rightarrow \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 6Ax = 4Bx$$

$$\frac{\partial \epsilon_{xy}}{\partial x} = 2Bxy + By^2 \Rightarrow \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} = 2Bx + 2By$$

$\epsilon_{xy} = Bx^2y + Bxy^2$

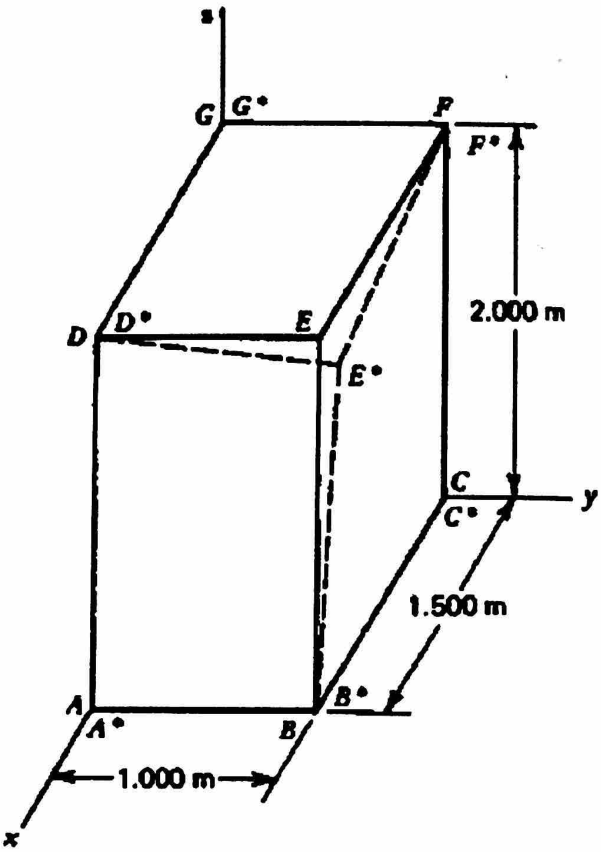
$$4Bx + 4By = 2(2Bx + 2By)$$

$$\Rightarrow 4B(x+y) = 4B(x+y)$$

✓ they satisfy compatibility Eqns

Example

Solid lines → undeformed
Dashed lines → Deformed



Displacement field

$u = C_1 xyz, \ v = C_2 xyz, \ w = C_3 xyz$

Location of E^* ⇒ $(1.504, 1.002, 1.996)$

Determine

① Constants C_1, C_2 and C_3

② Strain field at E

Solution

Deformed state E^*

Undeformed state E

Displacement = Deformed - Undeformed

① To find C_1, C_2 and C_3 we need to find deformations from

E to E^*

x direction $\quad E_x^* - E_x = u_E \Rightarrow u_E = 1.504 - 1.500 = 0.004$

y direction $\quad E_y^* - E_y = v_E \Rightarrow v_E = 1.002 - 1.000 = 0.002$

z direction $\quad E_z^* - E_z = w_E \Rightarrow w_E = 1.996 - 2.000 = -0.004$

At point E

$u_E = 0.004 = C_1(1.5)(1)(2) \Rightarrow C_1 = \dfrac{0.004}{3}$

$v_E = 0.002 = C_2(1.5)(1)(2) \Rightarrow C_2 = \dfrac{0.002}{3}$

$w_E = -0.004 = C_3(1.5)(1)(2) \Rightarrow C_3 = \dfrac{-0.004}{3}$

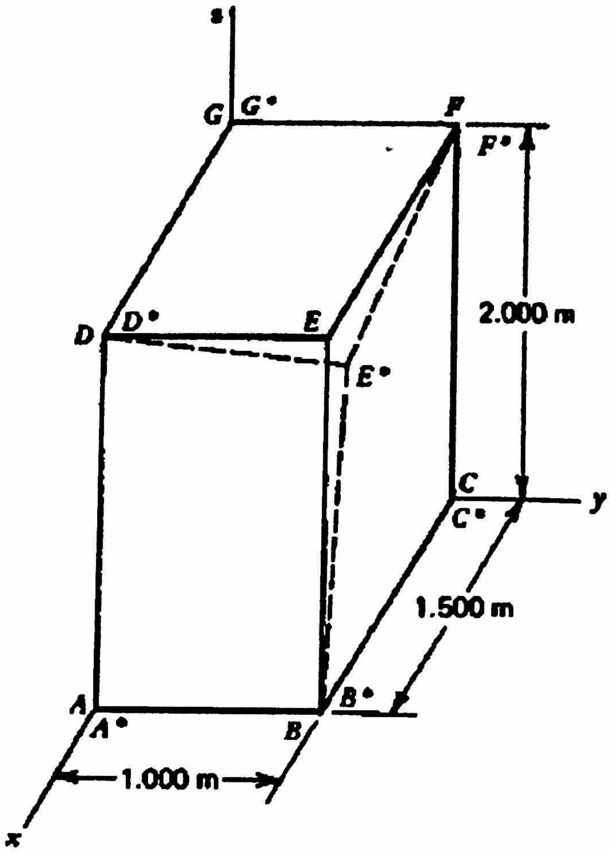
$$u = \frac{0.004}{3} xyz, \quad v = \frac{0.002}{3} xyz, \quad w = \frac{-0.004}{3} xyz$$

② Strain Field at E

$u = \frac{0.004}{3} xyz$, $v = \frac{0.002}{3} xyz$

$w = -\frac{0.004}{3} xyz$

$$[\epsilon] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}$$



$$\epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{0.004}{3} yz \quad , \quad y = 1, \ z = 2$$

$$\Rightarrow \boxed{\epsilon_{xx} = 0.00267}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = \frac{0.002}{3} xz \quad , \quad x = 1.5, \ z = 2$$

$$\boxed{\epsilon_{yy} = 0.002}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} = \frac{-0.004}{3} xy, \quad x = 1.5, \ y = 1$$

$$\boxed{\epsilon_{zz} = -0.002}$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$= \frac{1}{2}\left(\frac{0.002}{3} yz + \frac{0.004}{3} xz\right) \qquad x = 1, \ y = 1.5, \ z = 2$$

$$\epsilon_{xy} = 2.915 \times 10^{-3}$$

$$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}\left(\frac{-0.002}{3} yz + \frac{0.004}{3} xy\right) \qquad x = 1, \ y = 1.5, \ z = 2$$

$$\epsilon_{xz} = -3.5 \times 10^{-3}$$

$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \frac{1}{2}\left(\frac{-0.004}{3} xz + \frac{0.002}{3} xy\right) \qquad x = 1, \ y = 1.5, \ z = 2$$

$$\epsilon_{yz} = -1.5 \times 10^{-3}$$

$$[\epsilon] = \begin{bmatrix} 2.67 & 2.915 & -3.5 \\ 2.915 & 2 & -1.5 \\ -3.5 & 1.5 & -2 \end{bmatrix} \times 10^{-3}$$