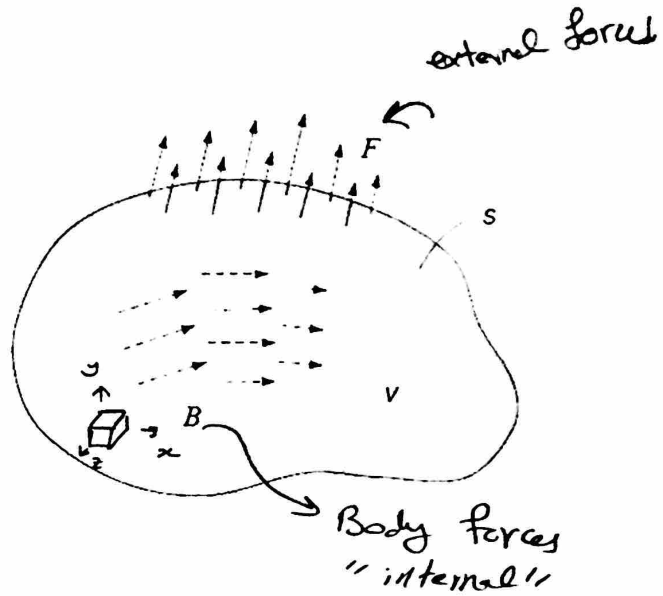
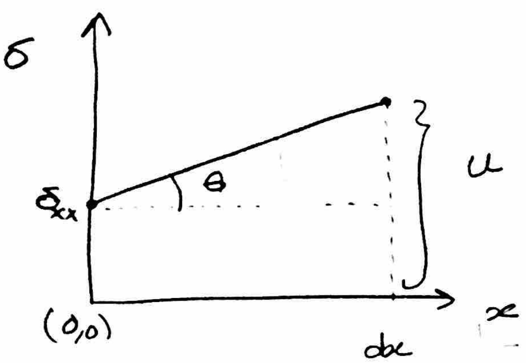
2.5 Differential equations of motion of a Deformable Body.

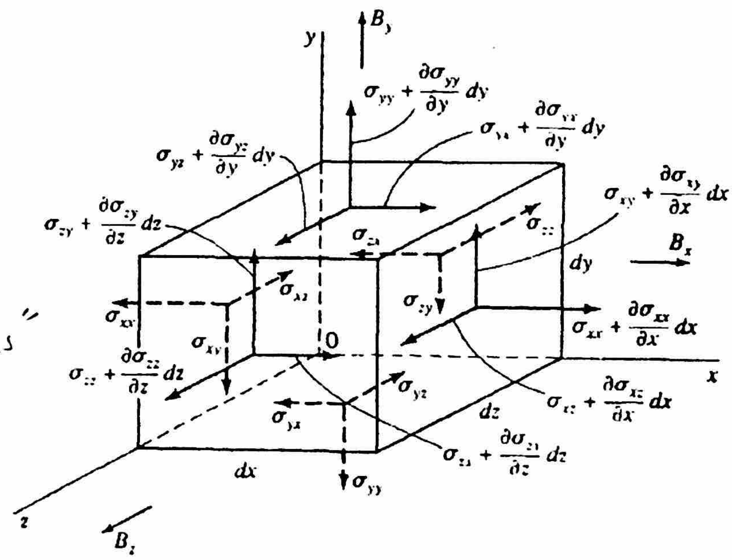
Assumptions

* For a deformed body stress changes with location (dx, dy, dz) — linearly.



* For example (σ_{xx})





$$\tan\theta = \frac{u - \sigma_{xx}}{dx} = \frac{\partial \sigma_{xx}}{\partial x}$$ "small deformations"

$$\Rightarrow \boxed{u = \sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} \cdot dx}$$

stress component on the other face (dx)

$$F = \sigma A, \quad B \left\langle \begin{matrix} B_x \\ B_y \\ B_z \end{matrix} \right.$$

* For equilibrium $\sum \text{Forces} = 0 \Rightarrow \sum F + \sum B = 0$

(1) $$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + B_x = 0$$

(2) $$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + B_y = 0$$

(3) $$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + B_z = 0$$

$$\frac{\partial \sigma_{ji}}{\partial j} + F_i = 0$$

Equilibrium equations

* We can transform Eq(1), Eq(2) and Eq(3) to Polar and Cylindrical coordinates. See text book Section 2.5 Eq (2.45) to Eq(2.54).

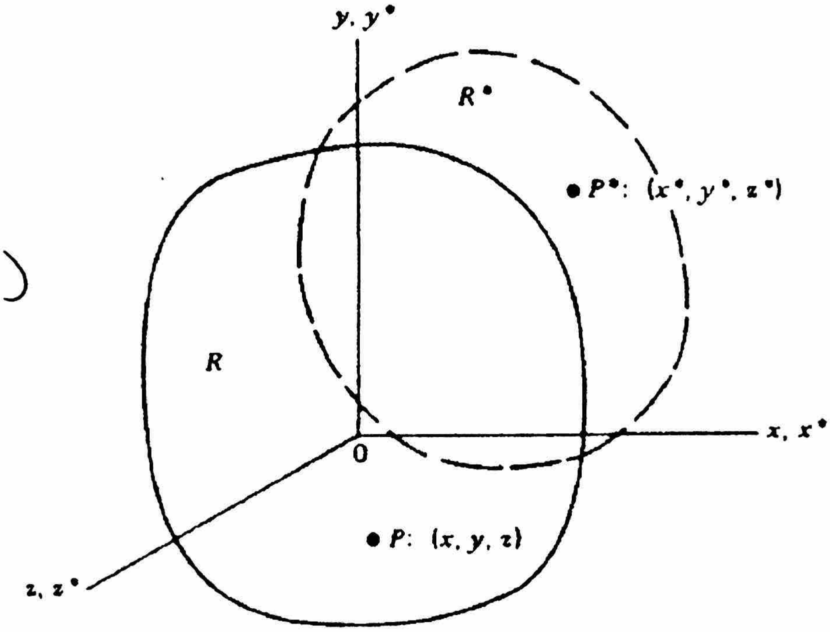
* In general in this course we will focus on the Cartesian coordinate system.

* Equilibrium equations mean that "each deformed body <u>must</u> satisfy eq(1), eq(2) and eq(3). otherwise, the body is <u>NOT</u> in equilibrium "Dynamic motion".

2.6 Deformation of a Deformable Body

* The unloaded and deformed body is enclosed in Region (R) coordinate system (x, y, z)

* The deformed body is enclosed in Region (R^*) coordinate system (x^*, y^*, z^*)



* For a point (P) in the undeformed state can be expressed in terms of (x, y, z).

- Also, can be written using (x^*, y^*, z^*) using coordinate transformation

* The point (P), becomes (P^*) in the deformed body and can be written in terms of (x^*, y^*, z^*)

  - Also, using coordinate transformation, can be written in terms of (x, y, z).

  - we can write

$$x^* = x + u$$
$$y^* = y + v$$
$$z^* = z + w$$

where u, v, w are the change of location of Point (P) to (P^*). $\Rightarrow$ Displacements (u, v, w)

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2.7 strain theory, transformation of strain and principal strains

* Undeformed state (R)
  ↳ points (P) and Q

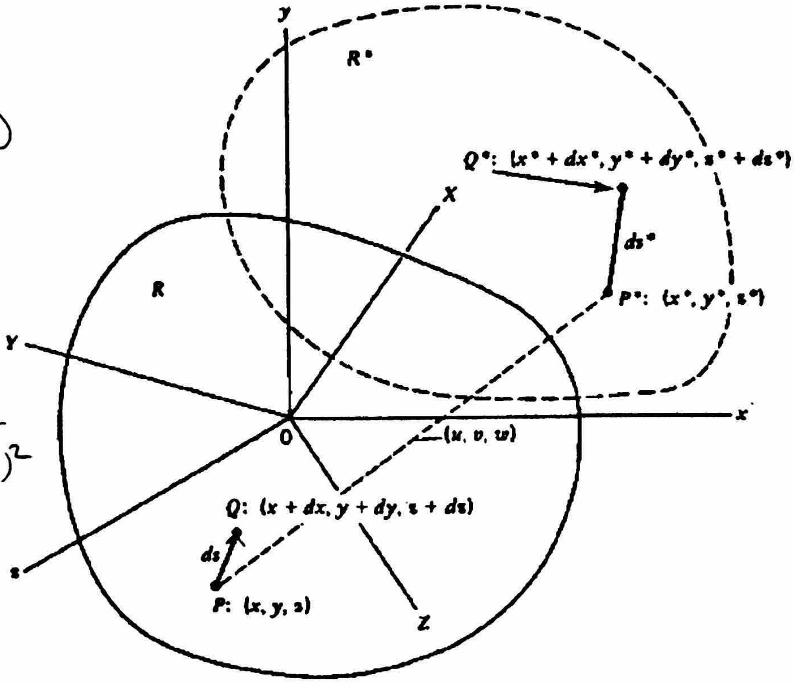
location of P (x, y, z)

location of Q $(x+dx, y+dy, z+dz)$

vector between P and Q is $\vec{ds}$

$$\vec{ds} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\text{length} \Rightarrow ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \qquad \text{Eq(1)}$$



* Deformed state (R^*)
  ↳ points (P^*) and (Q^*)

location of P^* (x^*, y^*, z^*)

location of Q^* $(x^* + dx^*, y^* + dy^*, z^* + dz^*)$

vector between P^* and Q^* is $\vec{ds^*}$

$$\vec{ds^*} = dx^*\,\hat{i} + dy^*\,\hat{j} + dz^*\,\hat{k}$$

$$\text{length} \Rightarrow ds^* = \sqrt{(dx^*)^2 + (dy^*)^2 + (dz^*)^2} \qquad - \text{Eq(2)}$$

* strain
$$\epsilon = \frac{\text{change of length}}{\text{original length}} = \frac{ds^* - ds}{ds} \qquad -③$$

Using coordinate transformation

$$\left.\begin{aligned}
dx^* &= \frac{\partial x^*}{\partial x}\cdot dx + \frac{\partial x^*}{\partial y}\cdot dy + \frac{\partial x^*}{\partial z}\cdot dz \\
dy^* &= \frac{\partial y^*}{\partial x}\,dx + \frac{\partial y^*}{\partial y}\cdot dy + \frac{\partial y^*}{\partial z}\cdot dz \\
dz^* &= \frac{\partial z^*}{\partial x}\cdot dx + \frac{\partial z^*}{\partial y}\,dy + \frac{\partial z^*}{\partial z}\cdot dz
\end{aligned}\right\} \quad \text{Eq(4)}$$

* Substitute Eq (1), Eq (2) in Eq (3) with the use of Eq (4) and u, v, w definitions.

$$\epsilon = l^2 \epsilon_{xx} + m^2 \epsilon_{yy} + n^2 \epsilon_{zz} + 2lm\, \epsilon_{xy} + 2ln\, \epsilon_{xz} + 2mn\, \epsilon_{yz}$$

where $l = \frac{dx}{ds}$, $m = \frac{dy}{ds}$, $n = \frac{dz}{dx}$

and strain components are:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$\epsilon_{xy} = \epsilon_{yx} = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right]$$

$$\epsilon_{xz} = \epsilon_{zx} = \frac{1}{2}\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial z}\right]$$

$$\epsilon_{yz} = \epsilon_{zy} = \frac{1}{2}\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial z}\right]$$

<u>For small strain theory</u> (Section 2.8)

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \quad \epsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \quad \epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

ϵ_{yx} $\quad\quad$ ϵ_{zy} $\quad\quad$ ϵ_{zx}

or, strain matrix

$$[\epsilon] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

For this matrix, we can

① Find strain transformation

② principal strains and principal directions

③ 2D and 3D Mohr's circle of strain.