Example

$$[\sigma] = \begin{bmatrix} 80 & 20 & 40 \\ 20 & 60 & 10 \\ 40 & 10 & 20 \end{bmatrix}$$

① Determine stress vector on aplane with a normal vector $\vec{R} = \hat{i} + 2\hat{j} + \hat{k}$

② Determine principal stresses (σ_1, σ_2 and σ_3), and τ_{max}.

③ Determine τ_{oct} and Compare to τ_{max}.

Solution

① stress on aplane $\Rightarrow$ $\vec{\sigma_p} = \sigma_{px}\hat{i} + \sigma_{py}\hat{j} + \sigma_{pz}\hat{k}$

$$\sigma_{px} = l\sigma_{xx} + m\sigma_{xy} + n\sigma_{xz}$$
$$\sigma_{py} = l\sigma_{xy} + m\sigma_{yy} + n\sigma_{yz}$$
$$\sigma_{pz} = l\sigma_{xz} + m\sigma_{yz} + n\sigma_{zz}$$

what are l, m, n? $\rightarrow$ Normalize $\vec{R}$

$$\text{Length}(R) = \sqrt{(1)^2 + (2)^2 + (1)^2} = \sqrt{6}$$

$$\vec{R_n} = \frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}$$

$$\Rightarrow \quad l = \frac{1}{\sqrt{6}}, \quad m = \frac{2}{\sqrt{6}}, \quad n = \frac{1}{\sqrt{6}}$$

$$\Rightarrow \sigma_{px} = \left(\frac{1}{\sqrt{6}}\right)(80) + \left(\frac{2}{\sqrt{6}}\right)(20) + \left(\frac{1}{\sqrt{6}}\right)(40) = 65.320$$

$$\sigma_{py} = \left(\frac{1}{\sqrt{6}}\right)(20) + \left(\frac{2}{\sqrt{6}}\right)(60) + \left(\frac{1}{\sqrt{6}}\right)(10) = 61.237$$

$$\sigma_{pz} = \left(\frac{1}{\sqrt{6}}\right)(40) + \left(\frac{2}{\sqrt{6}}\right)(10) + \left(\frac{1}{\sqrt{6}}\right)(20) = 32.660$$

$$\boxed{\vec{\sigma_p} = 65.320\hat{i} + 61.237\hat{j} + 32.660\hat{k}}$$

(2) Principal stresses

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 80 + 60 + 20 = 160$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz} - \sigma_{xy}^2 - \sigma_{xz}^2 - \sigma_{yz}^2$$

$$= (80)(60) + (80)(20) + (60)(20) - (20)^2 - (40)^2 - (10)^2$$

$$= 5500$$

$$I_3 = \det(\sigma) = 0$$

$$\Rightarrow \sigma^3 - 160\sigma^2 + 5500\sigma = 0$$

$$\sigma_1 = 110, \quad \sigma_2 = 50, \quad \sigma_3 = 0$$

τ_{max}

$$\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min}) = \frac{1}{2}(110 - 0) = 55$$

(3) τ_{oct}

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2}$$

$$= \frac{1}{3}\sqrt{(110-50)^2 + (110-0)^2 + (50-0)^2}$$

$$\cong 45$$

Compare τ_{oct} to τ_{max}

$\tau_{oct} < \tau_{max} \Rightarrow \tau_{max} = 1.223\,\tau_{oct}$

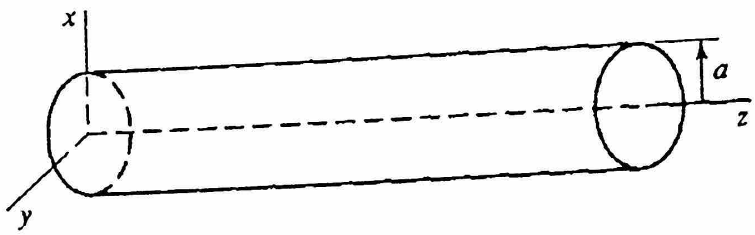
Example For a pure Torsion problem of the bar shown

$$[\sigma] = \begin{bmatrix} 0 & 0 & -Gy\beta \\ 0 & 0 & Gx\beta \\ -Gy\beta & Gx\beta & 0 \end{bmatrix}$$

G and β constants

Determine

- The principal stresses on lateral surface of the bar.
- The principal directions " " " " " ".



Solution

The lateral surface equation is an equation of a circle, as:

$$\boxed{a^2 = x^2 + y^2}$$

① principal stresses

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0$$

$$I_2 = \cancel{\sigma_{xx}\sigma_{yy}}^{0} + \cancel{\sigma_{xx}\sigma_{zz}}^{0} + \cancel{\sigma_{yy}\sigma_{zz}}^{0} - \cancel{\sigma_{xy}^2}^{0} - \sigma_{xz}^2 - \sigma_{yz}^2$$

$$= -\sigma_{xz}^2 - \sigma_{yz}^2 = -(-Gy\beta)^2 - (Gx\beta)^2$$

$$= -G^2y^2\beta^2 - G^2x^2\beta^2 = -G^2\beta^2(x^2+y^2)$$

$$= -G^2\beta^2a^2$$

$$I_3 = \det(\sigma) = 0$$

$$\Rightarrow \sigma^3 - G^2\beta^2 a^2 \sigma = 0 \Rightarrow \sigma(\sigma^2 - G^2\beta^2 a^2) = 0$$

$$\sigma = 0 , \quad \sigma^2 = G^2\beta^2 a^2 \Rightarrow \sigma = \pm G\beta a$$

$$\sigma_1 = G\beta a , \quad \sigma_2 = 0, \quad \sigma_3 = -G\beta a$$

② principal directions, as we did before

Dont forget to normalize

$$\vec{V_1} = -0.5\hat{i} + 0.5\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$$

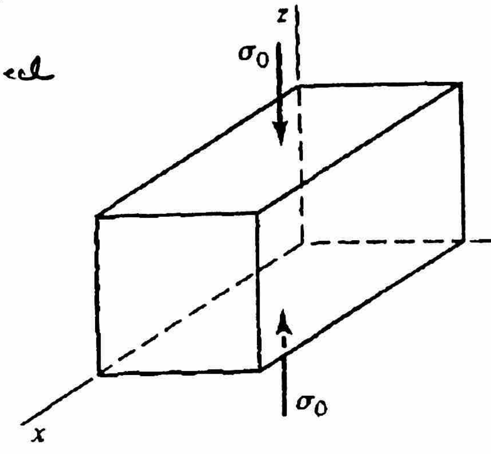
$$\vec{V_2} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

$$\vec{V_3} = \frac{1}{2}\hat{i} - 0.5\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$$

4 more details, See example 2.6 From text book

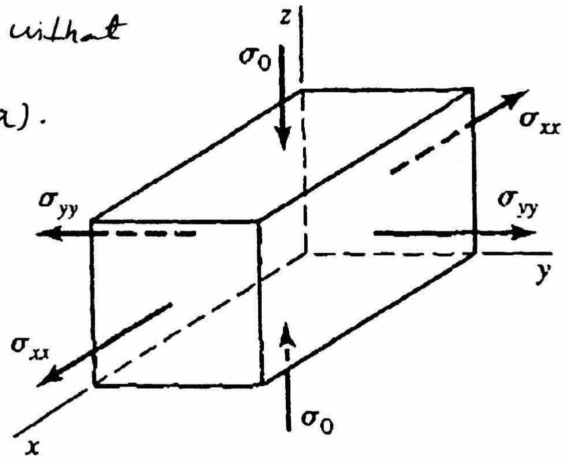
Example

(a)
For body and loading of Figure (a)
If Design specification says that the shear stress does not exceed 160 MPa, what is the maximum value of σ_0?



(a)

(b) For figure (b), If an extra σ_{xx} and σ_{yy} are added in tension. If $\sigma_{xx} > \sigma_{yy}$ .. Find the values of σ_{xx} and σ_{yy} If $\sigma_0 = 560$ MPa without exceeding design requirement of part (a).



(b)

(a) As shown, the body is principal stress state since that there is no shear stress

$\Rightarrow \sigma_1 = \sigma_2 = 0$ and $\sigma_3 = -\sigma_0$

Design requirement $\tau_{max} \leq 160$

$$\Rightarrow \tau_{max} = \frac{1}{2}(\sigma_{max} - \underbrace{\sigma_{min}}^{= \sigma_0}) \leq 160$$

$$= 0 - \sigma_0 \leq 320$$

$\Rightarrow$ Maximum allowable $\boxed{\sigma_0 = +320 \text{ MPa}}$

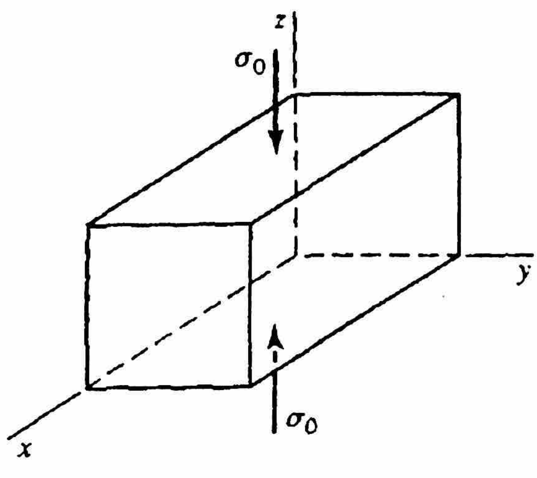
(b) we need to find $(\sigma_{xx}, \sigma_{yy})$ while $\tau_{max} \le 160$ MPa

and $\sigma_0 = 560$ MPa

Solution: we are in principal stress state, so:

$\sigma_1 = \sigma_{xx}$, $\sigma_2 = \sigma_{yy}$

$\sigma_3 = \sigma_0$



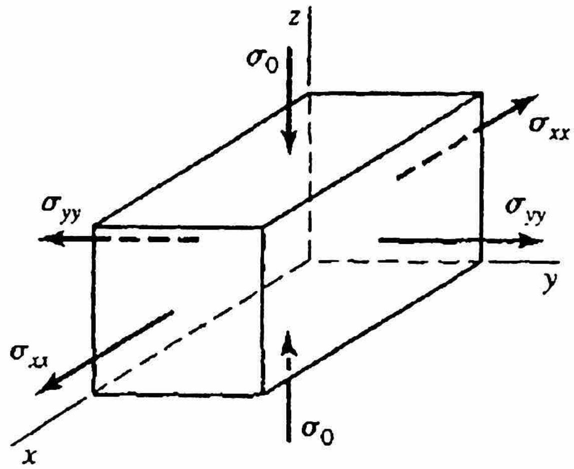
(a)

shear stresses, from 3D Mohr's circle

$$\tau_1 = \frac{1}{2}(\sigma_2 - \sigma_3)$$

$$\boxed{\tau_2 = \frac{1}{2}(\sigma_1 - \sigma_3)} \text{ — } \tau_{max}$$

$$\tau_3 = \frac{1}{2}(\sigma_1 - \sigma_2)$$



(b)

$$\tau_{max} = \frac{1}{2}(\sigma_{xx} - \sigma_0)$$

$$160 = \frac{1}{2}(\sigma_{xx} - {}^{-}560)$$

$$320 = \sigma_{xx} - {}^{-}560$$

$$\sigma_{xx} = -240 \text{ MPa}$$

let's $\tau_1 = \frac{1}{2}(\sigma_2 - \sigma_3) = \frac{1}{2}(\sigma_{yy} - \sigma_0)$

same as above

$$\sigma_{yy} = -240 \text{ MPa}$$