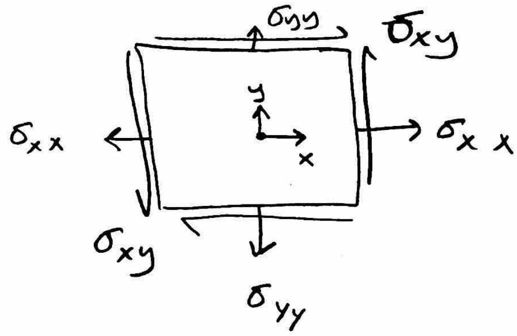
2.4.6 Plane stress

* For a 3D stress state
$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

* special case : Plane stress $\Rightarrow \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$

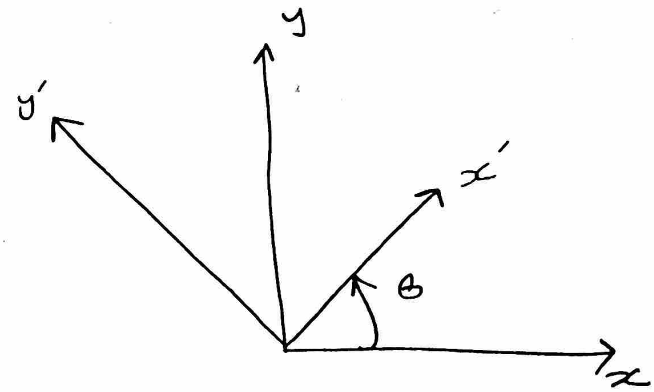


$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$$

* stress transformation

$[\sigma']$ is the transformed stress state

$$[\sigma'] = \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{xy} & \sigma'_{yy} \end{bmatrix}$$



as we discussed previously

$$[\sigma'] = [Q][\sigma][Q]^T \text{ — Eq(1)}, \quad [Q] \text{ transformation matrix}$$

$$[Q] = \begin{bmatrix} \cos(x', x) & \cos(x', y) \\ \cos(y', x) & \cos(y', y) \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos\frac{\pi}{2}-\theta \\ \cos\frac{\pi}{2}+\theta & \cos\theta \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{[Q]}$$

By evaluating Eq(1)

$$\sigma_{xx}' = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta \quad -a$$

$$\sigma_{yy}' = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta \quad -b$$

$$\sigma_{xy}' = -(\sigma_{xx} - \sigma_{yy})\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta) \quad -c$$

* If Eq(a) + Eq(b) $\Rightarrow$ $\sigma_{xx}' + \sigma_{yy}' = \sigma_{xx} + \sigma_{yy}$ $\quad -(d)$

keep in mind $\sigma_{xx}' \neq \sigma_{xx}$ and $\sigma_{yy}' \neq \sigma_{yy}$

* Use Eq(a), (b), (c) and (d) $\Rightarrow$ $\sigma_{xx}'\sigma_{yy}' - \sigma_{xy}'^2 = \sigma_{xx}\sigma_{yy} - \sigma_{xy}$ $\quad -(e)$

* In plane stress

stress invariants I_1, I_2 and I_3

$$I_1 = \sigma_{xx} + \sigma_{yy}$$

$$I_2 = \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2$$

$$I_3 = 0$$

2.4.7 Mohr's Circle in two dimensions

we can write Eq(a), (b) and (c), as:

$$\sigma'_{xx} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \sigma_{xy}\sin 2\theta \quad (A)$$

$$\sigma'_{yy} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta - \sigma_{xy}\sin 2\theta \quad (B)$$

$$\sigma'_{xy} = -\frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\theta + \sigma_{xy}\cos 2\theta \quad (C)$$

rewrite Eq(A) and Eq(C), let $\sigma_{Avg} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy})$

$$(\sigma'_{xx} - \sigma_{Avg})^2 = \Big(\underbrace{\tfrac{1}{2}(\sigma_{xx} - \sigma_{yy})}_{A}\cos 2\theta + \underbrace{\sigma_{xy}}_{B}\sin 2\theta\Big)^2 \quad +$$

$$(\sigma'_{xy})^2 = \Big(-\underbrace{\tfrac{1}{2}(\sigma_{xx} - \sigma_{yy})}_{A}\sin 2\theta + \underbrace{\sigma_{xy}}_{B}\cos 2\theta\Big)^2$$

$$(\sigma'_{xx} - \sigma_{Avg})^2 + (\sigma'_{xy})^2 = \frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2$$

this is an equation of a circle (Mohr's Circle)

$$\Rightarrow \quad (A\cos 2\theta + B\sin 2\theta)^2 = A^2\cos^2 2\theta + \cancel{2AB\sin 2\theta\cos 2\theta} + B^2\sin^2 2\theta \quad +$$

$$\Rightarrow \quad (-A\sin 2\theta + B\cos 2\theta)^2 = A^2\sin^2 2\theta - \cancel{2AB\sin 2\theta\cos 2\theta} + B^2\sin^2 2\theta$$

$$A^2(\sin^2 + \cos^2) + B^2(\sin^2 + \cos^2)$$

$$\Rightarrow A^2 + B^2$$

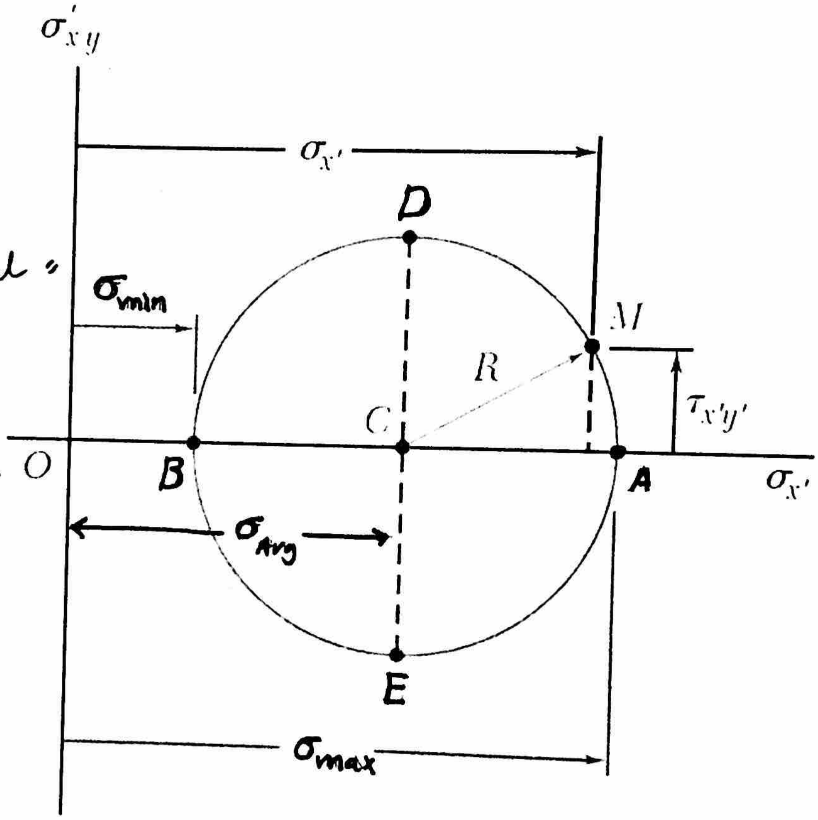
$$\Rightarrow \frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2$$

* The center of this circle is (C)

$$C = \sigma_{Avg} = \frac{\sigma_{xx} + \sigma_{yy}}{2}$$

* The Radius is (R)

$$R = \sqrt{\frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2}$$



At points A and B

Point A

$\sigma_1 = \sigma_{Avg} + R$ "Max Principal stress"

Point B

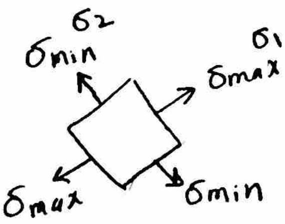
$\sigma_2 = \sigma_{Avg} - R$ "min principal stress"

At point D (or E)

$\tau = \tau_{max} = R$

Question ① What is θ that make $\sigma'_{xy} = 0$? θ_p

Answer $\Rightarrow$ Eq (C) $\Rightarrow$ $\sigma'_{xy} = -\frac{1}{2}(\sigma_{xx} - \sigma_{yy}) \sin 2\theta + \sigma_{xy} \cos 2\theta$



make $= 0$ $\Rightarrow$ $\tan 2\theta_p = \dfrac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}$

principal plane

It is the plane where $\sigma_{xy} = 0$

$\Rightarrow$ principal stress state

Question 2 what is θ makes max shear $\sigma'_{xy} = \tau_{max}$? θ_s

Answer

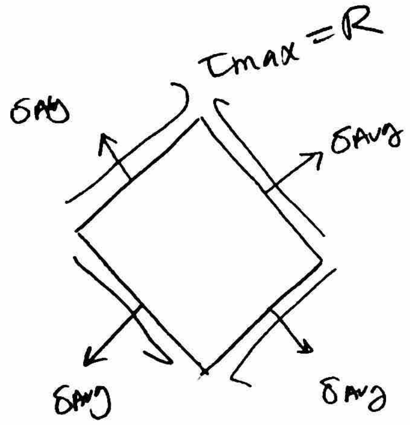
Eq(a) $\Rightarrow$ $\sigma'_{xx} = \sigma_{Avg} + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \sigma_{xy}\sin 2\theta$

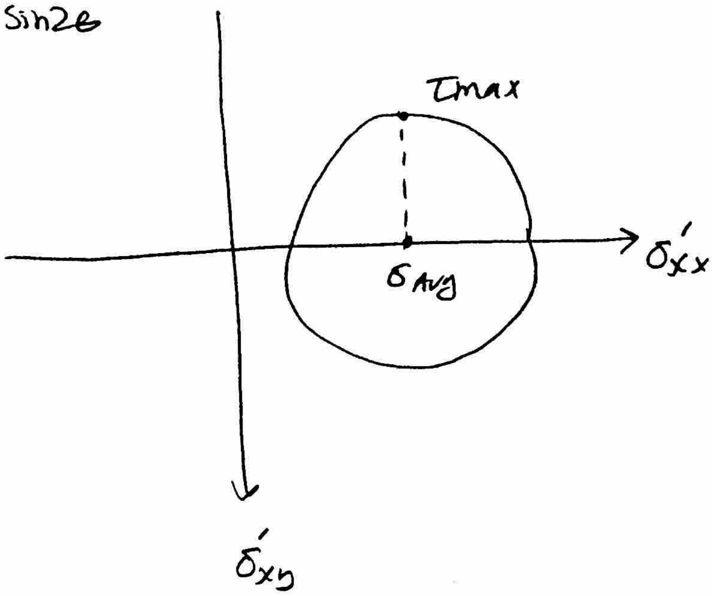
τ_{max} happens when $\sigma'_{xx} = \sigma_{Avg}$

so, $\cancel{\sigma_{Avg}} = \cancel{\sigma_{Avg}} + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \sigma_{xy}\sin 2\theta$

$$\Rightarrow \tan 2\theta_s = -\frac{\sigma_{xx} - \sigma_{yy}}{2\sigma_{xy}}$$

planes of max shear





Example $\sigma_{xx} = 180$, $\sigma_{yy} = 90$, $\sigma_{xy} = 50$

Find ① principal stresses using
(a) 2D Mohr's Circle
(b) 3D Mohr's Circle

② τ_{max}

Solution

① principal stresses

(a) 2D Mohr's Circle

$$\sigma_1 = \sigma_{Avg} + R \quad , \quad \sigma_2 = \sigma_{Avg} - R$$

$$\sigma_{Avg} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) = \frac{1}{2}(180 + 90) = 135$$

$$R = \sqrt{\frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} = \sqrt{\frac{1}{4}(180-90)^2 + (50)^2} = 67.27$$

$$\Rightarrow \sigma_1 = 202.27 \quad , \quad \sigma_2 = 67.73$$

(b) 3D Mohr's Circle

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0 \quad , \quad [\sigma] = \begin{bmatrix} 180 & 50 & 0 \\ 50 & 90 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$I_1 = 270, \quad I_2 = 13700, \quad I_3 = 0$$

$$\sigma^3 - 270\sigma^2 + 13700\sigma = 0 \Rightarrow \sigma(\sigma^2 - 270\sigma + 13700) = 0$$

$$\Rightarrow \sigma_1 = 202.27, \quad \sigma_2 = 67.73, \quad \sigma_3 = 0$$

(2) $$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{202.27 - 0}{2} = 101.135$$

To compute τ_{max}, we always use 3D Mohr's Circle