Mohr's Circle in three-dimensions

- For a stress state $[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$

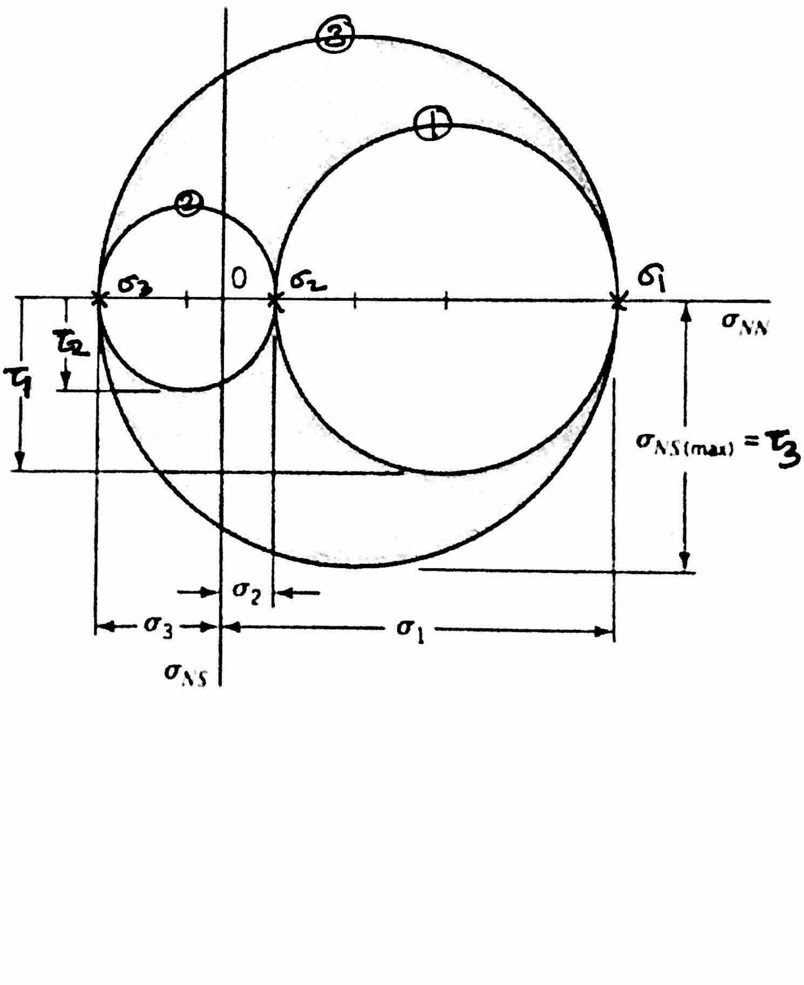
- The principal stresses are $\sigma_1 \geq \sigma_2 \geq \sigma_3$

- For each pair of principal stresses we will have a circle (σ_1, σ_2), (σ_1, σ_3) and (σ_2, σ_3)

For Circle #1 $\rightarrow (\sigma_1, \sigma_2)$

Center $C_1 = \dfrac{\sigma_1 + \sigma_2}{2}$

Radius $R_1 = \dfrac{\sigma_1 - \sigma_2}{2}$



For Circle #2 $\rightarrow (\sigma_2, \sigma_3)$

Center $C_2 = \dfrac{\sigma_2 + \sigma_3}{2}$

Radius $R_2 = \dfrac{\sigma_2 - \sigma_3}{2}$

For Circle #3 $\rightarrow (\sigma_1, \sigma_3)$

Center $C_3 = \dfrac{\sigma_1 + \sigma_3}{2}$

Radius $R_3 = \dfrac{\sigma_1 - \sigma_3}{2}$

Maximum Shear

τ_{max} = Radius of Circle 3

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}, \quad \tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2}$$

* For a plane (P) with a normal vector $\vec{V}$

$$\vec{V} = l\hat{i} + m\hat{j} + n\hat{k}$$, l, m, n are on principal coordinate system

* then, the normal stress on this plane (σ_{NN})

$$\sigma_{NN} = l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3 \quad - (1)$$

* the shear stress on this plane (σ_{NS})

$$\sigma_{NS} = \sqrt{l^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2 - \sigma_{NN}^2} \quad - (2)$$

* From previous lectures

$$l^2 + m^2 + n^2 = 1 \quad - (3)$$

Eq 1 $\Rightarrow$ $\sigma_{NN} = l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3$

Eq (2) $\Rightarrow$ $\sigma_{NS}^2 - \sigma_{NN}^2 = l^2\sigma_1^2 + m^2\sigma_2^2 + n\sigma_3^2$

Eq (3) $\Rightarrow$ $1 = l^2 + m^2 + n^2$

matrix Form

$$\underbrace{\begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \\ 1 & 1 & 1 \end{bmatrix}}_{[A]} \underbrace{\begin{Bmatrix} l^2 \\ m^2 \\ n^2 \end{Bmatrix}}_{\{x\}} = \underbrace{\begin{Bmatrix} \sigma_{NN} \\ \sigma_{NS}^2 - \sigma_{NN}^2 \\ 1 \end{Bmatrix}}_{\{b\}}$$

Solve for $\begin{Bmatrix} l^2 \\ m^2 \\ n^2 \end{Bmatrix}$ $\Rightarrow$ $\{x\} = \{b\}[A]^{-1}$

$$l^2 = \frac{\sigma_{NS}^2 + (\sigma_{NN} - \sigma_2)(\sigma_{NN} - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}$$

$$m^2 = \frac{\sigma_{NS}^2 + (\sigma_{NN} - \sigma_1)(\sigma_{NN} - \sigma_3)}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)}$$

$$n^2 = \frac{\sigma_{NS}^2 + (\sigma_{NN} - \sigma_1)(\sigma_{NN} - \sigma_2)}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)}$$

Example: $$[\sigma] = \begin{bmatrix} 120 & -55 & -75 \\ -55 & 55 & 33 \\ -75 & 33 & -85 \end{bmatrix}$$

① Find principal stresses

② Construct Mohr's Circle

③ Find Normal & shear stresses on

(a) plane A, $\vec{N}_A = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

(b) plane B, $\vec{N}_B = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

Plane A and B are relative to the principal axes.

Solution

① principal stresses

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = 90, \quad I_2 = -18{,}014, \quad I_3 = 471{,}680$$

$$\Rightarrow \sigma_1 = 176.80, \quad \sigma_2 = 24.06, \quad \sigma_3 = -110.86$$

② To draw Mohr's Circle we need Centers and Radius

$$C_1 = \frac{\sigma_1 + \sigma_2}{2} = \frac{176.80 + 24.06}{2} = 100.43$$

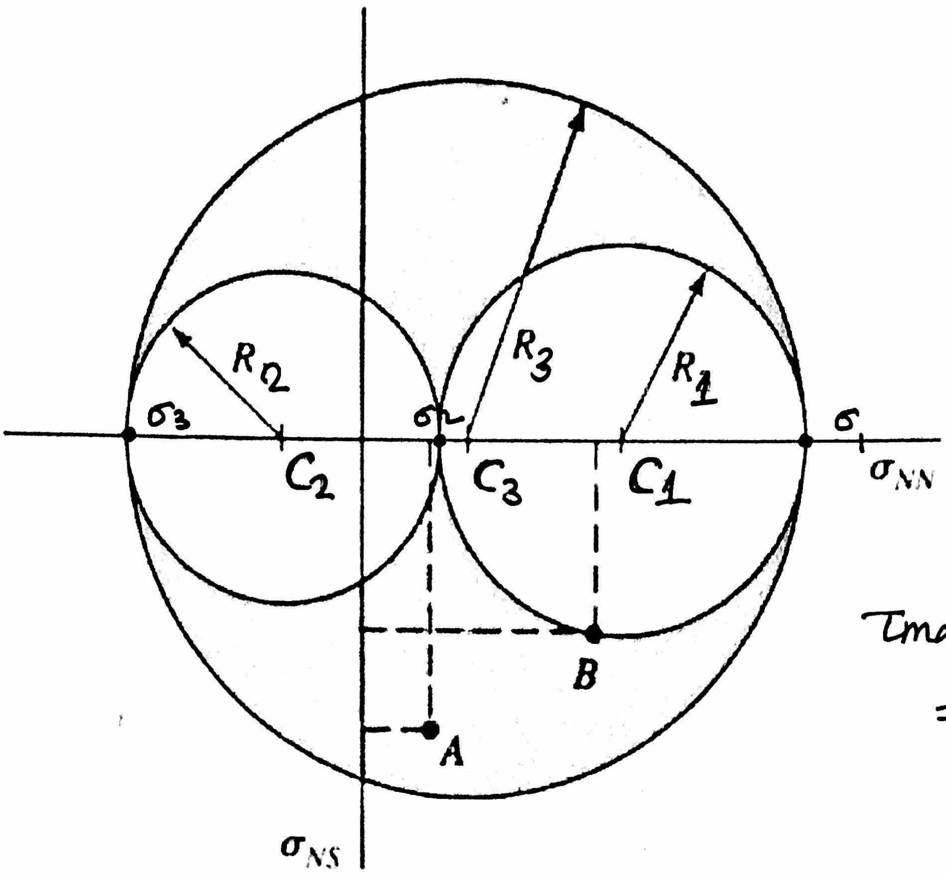
$$R_1 = \frac{\sigma_1 - \sigma_2}{2} = 76.37$$

$$C_2 = \frac{\sigma_2 + \sigma_3}{2} = \frac{24.06 + (-110.86)}{2} = -43.40$$

$$R_2 = \frac{\sigma_2 - \sigma_3}{2} = 67.46$$

$$C_3 = \frac{\sigma_1 + \sigma_3}{2} = \frac{176.80 + (-110.86)}{2} = 32.97$$

$$R_3 = \frac{\sigma_1 - \sigma_3}{2} = 143.83$$



$\tau_{max} = R_3$

$= 143.83$

(c) Normal & shear stress on plane (A) $l = m = n = \frac{1}{\sqrt{3}}$

Normal → $\sigma_{NNA} = l^2 \sigma_1 + m^2 \sigma_2 + n^2 \sigma_3 = \frac{1}{3}\left[(176.80) + 24.06 - 110.86\right]$

$= 30$

shear → $\sigma_{NSA} = \sqrt{l^2 \sigma_1^2 + m^2 \sigma_2^2 + n^2 \sigma_3^2 - \sigma_{NNA}^2}$

$= \sqrt{\frac{1}{3}(176.8)^2 + \frac{1}{3}(24.06)^2 + \frac{1}{3}(-110.86)^2 - (30)^2}$

$= 117.51$

Normal & shear stress on plane (B) $l = \frac{1}{\sqrt{2}}$, $m = \frac{1}{\sqrt{2}}$, $n = 0$

Normal $\sigma_{NNB} = 100.43$

Shear $\sigma_{NSB} = 76.37$