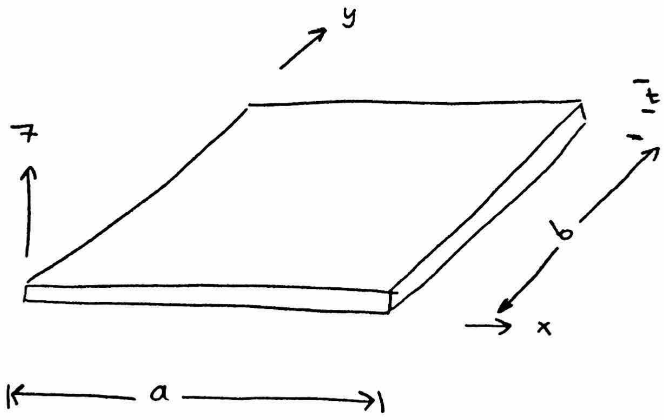
Applications to FDM

- Chapter 32.2 case study: Deflection of a plate (Page 917)

Plate deflection $w(x,y)$

Plate length (a), width (b)
thickness (t)

If distributed load $q(x,y)$



Equation of motion

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}, \quad D = \frac{E t^3}{12(1-\nu^2)}$$

E : Elastic modulus
ν : Poisson's ratio

How to solve 4th order PDE? → Transformation is required

let $$u(x,y) = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \quad — (1)$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} \quad (a)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \quad (b)$$

$$Eq(a) + Eq(b) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \quad (2)$$

Now, we have two 2nd order PDE's:

$$\left.\begin{aligned} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} &= u \\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{q}{D} \end{aligned}\right\}$$

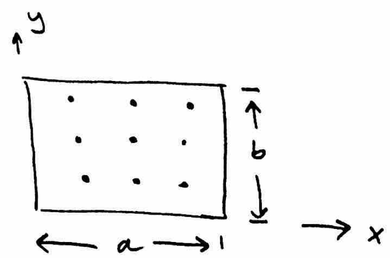
Two Elliptical PDE's can be solved easily

Using Centered Finite-differences

$$\frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{(\Delta x)^2} + \frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{(\Delta y)^2} = u_{i,j} \quad -(A)$$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} = \frac{q_{i,j}}{D} \quad -(B)$$

If $i = j = 3$ nodes



Eq(A) $\Rightarrow$ $[9\times9]\{9\times1\} = \{9\times1\}$ $\Rightarrow$ to Find $\underline{\underline{w}}$

Eq(B) $\Rightarrow$ $[9\times9]\{9\times1\} = \{9\times1\}$ $\Rightarrow$ Find u

we need Boundary conditions on all edges $\rightarrow$ 8 BC's (4 for x, 4 for y)

4th order PDE with 2 variables

For example

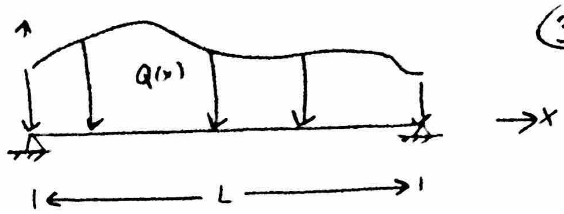
$$w(x,0) = w(x,b) = 0$$

$$w(0,y) = w(a,y) = 0$$

$$w_{xx}(x,0) = w_{xx}(x,b) = 0 \qquad w_{xx} = \frac{\partial^2 w}{\partial x^2}$$

$$w_{yy}(0,y) = w_{yy}(a,y) = 0 \qquad w_{yy} = \frac{\partial^2 w}{\partial y^2}$$

* Beam Deflection



Equation of motion

$$EI \frac{d^4 w}{dx^4} = Q(x)$$

$w(x)$: Beam deflection

E : Elastic modulus , $Q(x)$: Loading

I : moment of Inertia

4th order ODE

→ we need transformation

let $u = \frac{d^2 w}{dx^2} \Rightarrow \frac{d^2 u}{dx^2} = \frac{d^4 w}{dx^4} = \frac{Q(x)}{EI}$

Now, we have two 2nd order ODE's :

$$\left.\begin{aligned} \frac{d^2 w}{dx^2} &= u \\ \frac{d^2 u}{dx^2} &= \frac{Q(x)}{EI} \end{aligned}\right\} \text{ we can solve using F.D.M}$$

Using centered finite-differences

$$\frac{w(x_{i+1}) - 2w(x_i) + w(x_{i-1})}{(\Delta x)^2} = u(x_i)$$

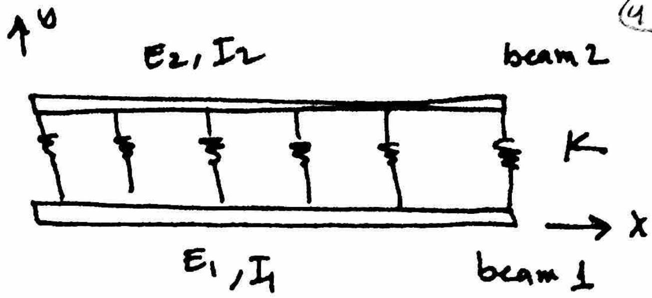
$$\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{(\Delta x)^2} = \frac{Q(x_i)}{EI}$$

We need 4 Boundary conditions

$w(0) = w(L) = 0$

$w_{xx}(0) = w_{xx}(L) = 0$, $w_{xx} = \frac{d^2 w}{dx^2}$

* Two Coupled beams



Equation of motion

$$E_1 I_1 \frac{d^4 w_1}{dx^4} = K\,\delta(x)$$

$$E_2 I_2 \frac{d^4 w_2}{dx^4} = -K\,\delta(x)$$

$\circ\ \delta(x) = w_2(x) - w_1(x)$

E_1, E_2 Elastic modulus of beam 1 and beam 2

I_1, I_2 moment of Inertia of beam 1 and beam 2

w_1 and w_2 beam 1 and beam 2 deflections

$\delta(x)$ springs deflection

$$\delta(x) = w_2(x) - w_1(x) \Rightarrow \frac{d^4\delta}{dx^4} = \frac{d^4 w_2}{dx^4} - \frac{d^4 w_1}{dx^4}$$

$$\begin{cases} \dfrac{d^4 w_1}{dx^4} = \dfrac{K}{E_1 I_1}\delta(x) \\ \dfrac{d^4 w_2}{dx^4} = \dfrac{-K}{E_2 I_2}\delta(x) \end{cases}$$

$$\Rightarrow \frac{d^4\delta}{dx^4} = -K\,\delta(x)\left[\frac{1}{E_2 I_2} + \frac{1}{E_1 I_1}\right] \qquad \frac{1}{D} = \left[\frac{1}{E_1 I_1} + \frac{1}{E_2 I_2}\right]$$

$$\Rightarrow \frac{d^4\delta}{dx^4} + \frac{K}{D}\delta(x) = 0\ , \quad 4\alpha^4 = \frac{K}{D}$$

$$\frac{d^4\delta}{dx^4} + 4\alpha^4\,\delta(x) = 0$$

$\rightarrow$ Now, we Solve this equation

$$u(x) = \frac{d^2\delta}{dx^2} \Rightarrow \frac{d^2 u}{dx^2} = \frac{d^4\delta}{dx^4} = -4\alpha^4\,\delta(x)$$

Now, we have two 2nd order ODE's

(1) $u(x) = \dfrac{d^2\delta}{dx^2}$

(2) $-4\alpha^4\delta = \dfrac{d^2 u}{dx^2}$

those equations can be Solved as before

we need 8 Boundary conditions $\rightarrow$ 4 beam 1, 4 beam 2