4.7 Lagrange's Equations

If we remember from chapter one, for a system with kinetic energy (T) and potential energy (V), then the Lagrangian (L) can be defined as ($L = T - V$) then the equation of motion of SDOF can be written as:-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 ,$$

$q, \dot{q}$ → x and $\dot{x}$ (spring-mass)
→ θ and $\dot{\theta}$ (Pendulum and Torsional)

For MDOF Systems:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$i = 1, 2, \ldots,$ number of DOF

$q_i, \dot{q}_i$ → $x_i, \dot{x}_i$
→ $\theta_i, \dot{\theta}_i$

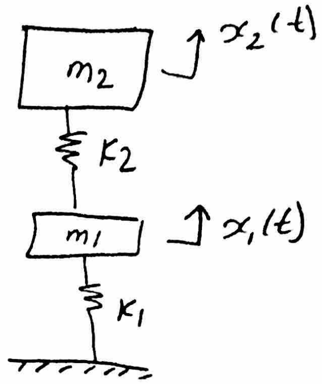
Systems undamped and Free vibration (unforced)

However, for Damped/Forced MDOF systems:-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

, Q_i: damping and Forcing terms

Example: Find EOM using Lagrange method.



$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = \frac{1}{2} K_1 x_1^2 + \frac{1}{2} K_2 (x_2 - x_1)^2$$

Solution

Lagrange eq'n: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (Q_i = 0)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

$$L = T - V = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} K_1 x_1^2 - \frac{1}{2} K_2 (x_2 - x_1)^2$$

$\frac{\partial L}{\partial \dot{x}_i}$

$\xrightarrow{i=1} \frac{\partial L}{\partial \dot{x}_1} = m_1 \dot{x}_1 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$

$\xrightarrow{i=2} \frac{\partial L}{\partial \dot{x}_2} = m_2 \dot{x}_2 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2$

$\frac{\partial L}{\partial x_i}$

$\xrightarrow{i=1} \frac{\partial L}{\partial x_1} = -K_1 x_1 - (-K_2 (x_2 - x_1)) = -K_1 x_1 + K_2 (x_2 - x_1)$

$\xrightarrow{i=2} \frac{\partial L}{\partial x_2} = -K_2 (x_2 - x_1)$

$\underline{i=1}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \Rightarrow m_1 \ddot{x}_1 - \left[-K_1 x_1 + K_2 (x_2 - x_1)\right] = 0$$

$$m_1 \ddot{x}_1 + (K_1 + K_2) x_1 - K_2 x_2 = 0 \qquad \text{—(1)}$$

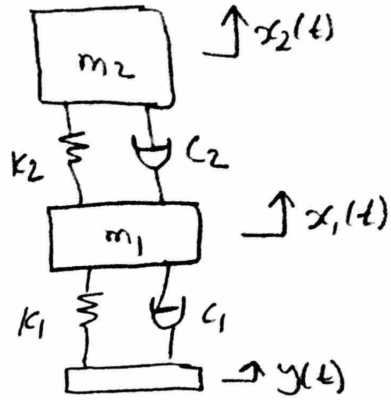
$\underline{i=2}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \Rightarrow m_2 \ddot{x}_2 - \left(-K_2 (x_2 - x_1)\right) = 0$$

$$m_2 \ddot{x}_2 - K_2 x_1 + K_2 x_2 = 0$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Example: For the 2DOF system with base excitation, Find EOM using Lagrange method



$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = \frac{1}{2} k_1 (x_1 - y)^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

$$Q_i = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} -c_1(\dot{x}_1 - \dot{y}) + c_2(\dot{x}_2 - \dot{x}_1) \\ -c_2(\dot{x}_2 - \dot{x}_1) \end{Bmatrix}$$

Solution

Lagrange eq'n: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = Q_i$, $L = T - V \Rightarrow$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}^2 - \frac{1}{2} k_1 (x_1 - y)^2 - \frac{1}{2} k_2 (x_2 - x_1)^2$$

$\frac{\partial L}{\partial \dot{x}_i}$

$\xrightarrow{i=1}$ $\frac{\partial L}{\partial \dot{x}_1} = m_1 \dot{x}_1 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$

$\xrightarrow{i=2}$ $\frac{\partial L}{\partial \dot{x}_2} = m_2 \dot{x}_2 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2$

$\frac{\partial L}{\partial x_i}$

$\xrightarrow{i=1}$ $\frac{\partial L}{\partial x_1} = -k_1(x_1 - y) + k_2(x_2 - x_1)$

$\xrightarrow{i=2}$ $\frac{\partial L}{\partial x_2} = -k_2(x_2 - x_1)$

Q_i

$\xrightarrow{i=1}$ $Q_1 = -c_1(\dot{x}_1 - \dot{y}) + c_2(\dot{x}_2 - \dot{x}_1)$

$\xrightarrow{i=2}$ $Q_2 = -c_2(\dot{x}_2 - \dot{x}_1)$

$\underline{i=1} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = Q_1$

$$\Rightarrow m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 + (c_1 + c_2)\dot{x}_1 - c_2 x_2 = k_1 y + c_1 \dot{y} \quad (1)$$

$\underline{i=2} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = Q_2$

$$\Rightarrow m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 + c_2 \dot{x}_2 - c_2 \dot{x}_1 = 0$$

Matrix Form

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

$$F_1 = k_1 y + c_1 \dot{y}$$