1.4 Modeling and energy methods

For a system with a kinetic energy (T) and potential energy (U), The Lagrangian of this system (L) is defined as $L = T - U$. Therefore, the Equation of motion of this system is given by:-

Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

$q, \dot{q} = x, \dot{x}$ spring-mass

$q, \dot{q} = \theta, \dot{\theta}$ Pendulum, Torsional system.

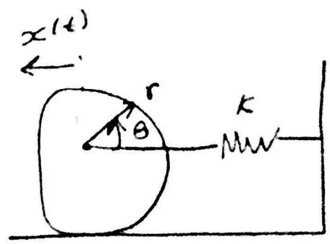
- For this system, the natural frequency can be obtain from

$$T_{max} = U_{max}$$

Example

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \frac{J}{r^2} \dot{x}^2$$

$$U = \frac{1}{2} k x^2$$



- Find nat Frequency of this system

Solution

At Nat. Freq

$$T_{max} = U_{max} \Rightarrow \frac{1}{2} m \dot{x}^2_{max} + \frac{1}{2} \frac{J}{r^2} \dot{x}^2_{max} = \frac{1}{2} k x^2_{max}$$

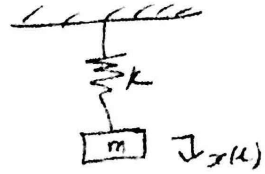
$$x(t) = X \sin(\omega_n t + \phi) \rightarrow x(t)_{max} = X$$

$$\dot{x}(t) = \omega_n X \cos(\omega_n t + \phi) \Rightarrow \dot{x}(t) = V = \omega_n X$$

$$\frac{1}{2} m (\omega_n X)^2 + \frac{1}{2} \frac{J}{r^2} (\omega_n X)^2 = \frac{1}{2} k (X^2)$$

$$\Rightarrow \omega_n^2 = \frac{\frac{1}{2} k}{\frac{1}{2} m + \frac{1}{2} \frac{J}{r^2}} \Rightarrow \omega_n = \sqrt{\frac{k}{m + J/r^2}}$$

Example Spring-mass system



$$T = \frac{1}{2} m \dot{x}^2$$

$$U = \frac{1}{2} k x^2$$

Find Eq. of motion

Solution

$$L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \quad , \quad q = x, \quad \dot{q} = \dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \Rightarrow \frac{d}{dt} = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = -kx$$

$$\Rightarrow \quad m\ddot{x} - (-kx) = 0 \quad \Rightarrow \boxed{m\ddot{x} + kx = 0}$$

Nat. Freq

$$x_{max} = X$$

$$\dot{x}_{max} = w_n X$$

$$T_{max} = U_{max} \quad \Rightarrow \quad \cancel{\tfrac{1}{2}} m \dot{x}^2_{max} = \cancel{\tfrac{1}{2}} k x^2_{max}$$

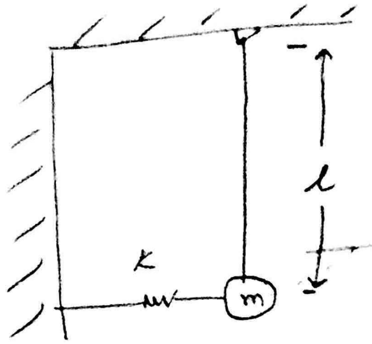
$$m w_n^2 \cancel{X^2} = k \cancel{X^2} \Rightarrow w_n^2 = \frac{k}{m} \Rightarrow w_n = \sqrt{\frac{k}{m}}$$

Example

$$T = \frac{1}{2} J\dot{\theta}^2 = \frac{1}{2} ml^2 \dot{\theta}^2$$

$$U = \frac{1}{2} kx^2 + mgh$$

Find Equation of motion

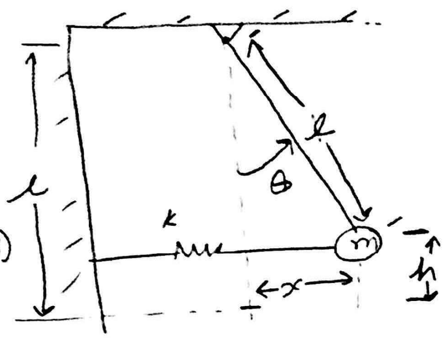


Solution

$$h = l - l\cos\theta$$
$$h = l(1-\cos\theta)$$

$$x = l\sin\theta$$

$$\Rightarrow U = \frac{1}{2} k l^2 \sin^2\theta + mgl(1-\cos\theta)$$



$$L = T - U$$
$$= \frac{1}{2} ml^2 \dot{\theta}^2 - \frac{1}{2} k l^2 \sin^2\theta - mgl(1-\cos\theta)$$

$$\Rightarrow mgl + mgl\cos\theta$$

Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \Rightarrow \frac{d}{dt} = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -kl^2 \sin\theta\cos\theta - mgl\sin\theta$$

$$ml^2\ddot{\theta} - \left(-kl^2 \sin\theta\cos\theta - mg\cancel{l}\sin\theta\right) = 0$$

$$ml\ddot{\theta} + kl\sin\theta\cos\theta + mg\sin\theta = 0$$

For small $\theta \Rightarrow$ $\sin\theta = \theta$, $\cos\theta = 1$

$$ml\ddot{\theta} + kl\theta + mg\theta = 0$$

$$\Rightarrow \ddot{\theta} + \theta\frac{(kl+mg)}{ml} = 0$$

$$w_n^2 = \frac{kl+mg}{ml} \Rightarrow w_n = \sqrt{\frac{kl+mg}{ml}}$$