(6)

2.2.4 Forced Vibrations of Undamped MDOF System

EOM

$$[m]\{\ddot{x}\} + [k]\{x\} = \{f(t)\}$$

Normalized modal matrix

$$\{x(t)\} = [U_n]\,\eta_i(t) \quad , \quad \eta_i(t) = A_i \cos\omega_i t + B_i \sin\omega_i t$$

Substitute in EOM and premultiply by $[U_n]^T$

$$\underbrace{[U_n]^T[m][U_n]}_{[I]}\ddot{\eta}_i + \underbrace{[U_n]^T[k][U_n]}_{[\omega_i^2]}\{x\} = \underbrace{[U_n]^T\{f(t)\}}_{Q_i(t) = \{Q(t)\} = \begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_n(t) \end{Bmatrix}}$$

$$\ddot{\eta}_i + \omega_i^2 \eta_i = Q_i(t)$$

we can use the convolution Integral to find $\eta_i(t)$

$$\eta_i(t) = \int_0^t Q_i(\tau)\,h(t-\tau)\,d\tau + g_i(t)\,\eta_i(0) + h_i(t)\,\dot{\eta}_i(0)$$

Initial conditions.

Remember.

$$\eta_i(0) = [U_n]^T[m]\{x(0)\}$$

$$\dot{\eta}_i(t) = [U_n]^T[m]\{\dot{x}(0)\}$$

$$g_i(t) = \cos\omega_i t$$

$$h_i(t) = \frac{1}{\omega_i}\sin\omega_i t$$

2.2.5 Forced vibration of damped MDOF systems

EOM

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [K]\{x\} = \{f(t)\}$$

For proportional damping

$$[c] = \alpha [m] + \beta [k]$$

function of mass and stiffness matrices.

α, β constants

$$\Rightarrow [m]\{\ddot{x}\} + (\alpha[m] + \beta[K])\{\dot{x}\} + [K]\{x\} = \{f(t)\} \quad \text{—}(*)$$

Solution

$$\{x(t)\} = [U_n]\eta_i(t)$$

$[U_n] \rightarrow$ Normalized modal matrix

$$\eta_i(t) = A_i \cos w_i t + B_i \sin w_i t$$

Substitute in Eq(*) and Premultiply by $[U_n]^T$.

$$\Rightarrow \underbrace{[U_n]^T[m][U_n]}_{I}\ddot{\eta}_i + (\alpha \underbrace{[U_n]^T[m][U_n]}_{I} + \beta \underbrace{[U_n]^T[K][U_n]}_{[w_i^2]})\dot{\eta}_i + \underbrace{[U_n]^T[K][U_n]}_{[w_i^2]}\eta_i = \underbrace{[U_n]^T\{f\}}_{Q_i}$$

$$\ddot{\eta}_i + (\alpha[I] + \beta[w_i^2])\dot{\eta}_i + [w_i^2]\eta_i = Q_i(t)$$

Let $2\zeta_i w_i = \alpha + \beta w_i \qquad i = 1, 2, \ldots, n$

$$\ddot{\eta}_i + 2\zeta_i w_i \dot{\eta}_i + w_i^2 \eta_i = Q_i(t)$$

Remember

$$\eta_i(0) = [U_n]^T[m]\{x(0)\}$$

$$\dot{\eta}_i(0) = [U_n]^T[m]\{\dot{x}(0)\}$$

$$\eta_i(t) = \int_0^t Q_i(\tau) h(t-\tau)d\tau + g_i(t)\,\eta_i(0) + h_i(t)\,\dot{\eta}_i(0)$$

$$g_i(t) = e^{\zeta_i w_i t}\left(\cos w_{d_i} t + \frac{\zeta_i w_i}{w_{d_i}} \sin w_{d_i} t\right), \quad h_i(t) = \frac{1}{w_{d_i}} e^{-\zeta_i w_i t} \sin w_{d_i} t$$

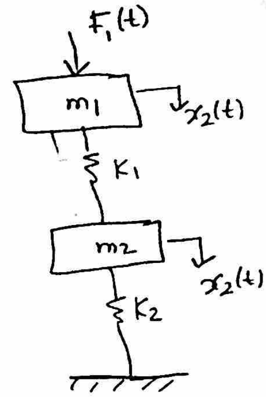
$$w_{d_i} = w_i\sqrt{1-\zeta_i^2}$$

Example $m_1 = 200\,Mg$ $\quad k_1 = 150\ MN/m$

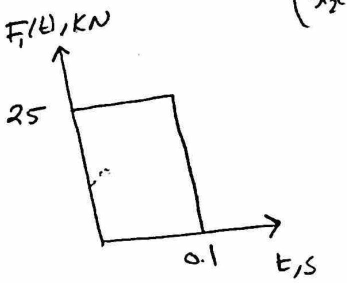
$m_2 = 250\,Mg$ $\quad k_2 = 75\ MN/m$

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



$$[m]\{\ddot{x}\} + [k]\{x\} = \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix}$$



$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$[k] = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix}$$

$$F_1(t) = \begin{cases} 25\,kN & 0 \le t \le 0.1\ sec \\ 0 & t > 0.1\ sec \end{cases}$$

Find ① Natural Frequencies and modeshapes

② Express responses (x_1 and x_2) in terms of Convolution Integral

solution

$\{x\} = \{X\} \sin(\omega t + \phi)$

For natural frequency and mode shape calculations, we always drop damping and Forces. Just keep [m] and [k]

$$\Rightarrow \left[[k] - \omega^2 [m]\right]\{X\} = 0$$

det = 0

$$\left| \begin{bmatrix} 150 & -150 \\ -150 & 225 \end{bmatrix} - \omega^2 \begin{bmatrix} 200 & 0 \\ 0 & 250 \end{bmatrix} \right| \times 10^6 = 0$$

kg

$$\begin{vmatrix} 150 - \omega^2 0.2 & -150 \\ -150 & 225 - \omega^2 0.25 \end{vmatrix} = (150 - \omega^2 0.2)(225 - \omega^2 0.25) - (-150)(-150) = 0$$

$$\Rightarrow \omega_1^2 = 150\ (rad/s)^2 \qquad \omega_2^2 = (1500)\ (rad/sec)^2$$

$\omega_1 = 12.2474$ rad/s, $\omega_2 = 38.7298$ rad/sec

You can use MATLAB $[V, D] = eig(k, m)$

Mode Shapes $\quad \omega_i^2$

Cont'd ④

To Find modeshapes $\{X\}$

$\omega_1^2 = 150$

$$\left[[K] - \omega_1^2 [m]\right]\{X^{(1)}\} = 0 \Rightarrow \begin{bmatrix} 120 & -150 \\ -150 & 187.5 \end{bmatrix} \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = 0$$

$$120 X_1^{(1)} - 150 X_2^{(1)} = 0$$

$$X_1^{(1)} = \frac{150}{120} X_2^{(2)} \Rightarrow X_1^{(1)} = 1.25 X_2^{(1)}$$

$$-150 X_1^{(1)} + 187.5 X_2^{(1)} = 0$$

$$X_1^{(1)} = \frac{187.5}{150} X_2^{(1)} \Rightarrow X_1^{(1)} = 1.25 X_2^{(1)}$$

$\Rightarrow$ Same Equation $\Rightarrow X_2^{(1)} = 1$

$$\{X^{(1)}\} = \begin{Bmatrix} 1.25 \\ 1 \end{Bmatrix}$$

$\omega_2^2 = 1500$

$$\begin{bmatrix} -150 & -150 \\ -150 & -150 \end{bmatrix} \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow X_1^{(2)} = -X_2^{(2)} \Rightarrow \text{let } X_2^{(2)} = 1 \Rightarrow \{X^{(2)}\} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\Rightarrow [U] = \left[\{X^{(1)}\} \quad \{X^{(2)}\}\right] = \begin{bmatrix} 1.25 & -1 \\ 1 & 1 \end{bmatrix}$$

To normalize $[U] \rightarrow [U_n]$

$$[U]^T [m][U] = \begin{bmatrix} \overset{a}{0.5625} & 0 \\ 0 & \underset{b}{0.45} \end{bmatrix} \Rightarrow [U_n] = \begin{bmatrix} \frac{1.25}{\sqrt{a}} & \frac{-1}{\sqrt{b}} \\ \frac{1}{\sqrt{a}} & \frac{1}{\sqrt{b}} \end{bmatrix}$$

$$[U_n] = \begin{bmatrix} 1.667 & -1.4907 \\ 1.333 & 1.4907 \end{bmatrix}$$

Cont'd

$$\eta_i(t) = \int_0^t Q_i(\tau)\, h(t-\tau)\, d\tau + g(t)\cancel{\eta_i(0)}^{0} + h(t)\cancel{\dot{\eta}_i(0)}^{0}$$

$$[U_n]^T[m][U_n]\ddot{\eta}_i(t) + [U_n]^T[k][U_n]\eta_i(t) = [U_n]^T\{F(t)\} \quad Q_i(t)$$

$0 \le t \le 0.1$

$$\eta_i(t) = \int_0^t Q_i(\tau)\, h(t-\tau)\, d\tau \quad , i = 1,2$$

$t > 0.1$

$$\eta_i(t) = \int_0^{0.1} Q_i(\tau)\, h(t-\tau)\, d\tau + \int_{0.1}^t \cancel{Q_i(\tau)}^{0}\, h(t-\tau)\, d\tau \quad , i = 1,2$$

$$Q_i = \begin{Bmatrix} Q_1(t) \\ Q_2(t) \end{Bmatrix} = \begin{bmatrix} 1.667 & -1.4907 \\ 1.333 & +1.4907 \end{bmatrix} \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix}$$

$$Q_1(t) = 1.667\, F_1(t)$$

$$Q_2(t) = 1.333\, F_1(t)$$

$$\{x(t)\} = [U_n]\,\eta_i(t) \quad , i = 1,2$$

Response for each mass